Resistência dos Materiais II

Flexão
Parte III

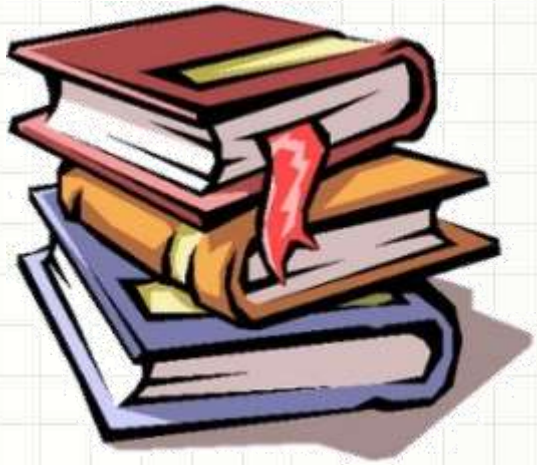
Prof. Dr. Daniel Caetano

2018 - 2

Objetivos

- Conceituar a flexo-compressão
- Conceituar e determinar o núcleo central de inércia
- Conceituar a flexão assimétrica
- Conceituar a flexão oblíqua
- Determinar a posição da linha neutra em barras sob flexão pura oblíqua

Material de Estudo



Material	Acesso ao Material
Apresentação	http://www.caetano.eng.br/ (Resistência dos Materiais II – Aula 11)
Material Didático	Resistência dos Materiais (Hibbeler), págs 216 a 224 e 304 a 320.
Biblioteca Virtual	“Resistência dos Materiais”

REVENDO...

Flexão Pura Reta

- Pode-se calcular σ a partir de M

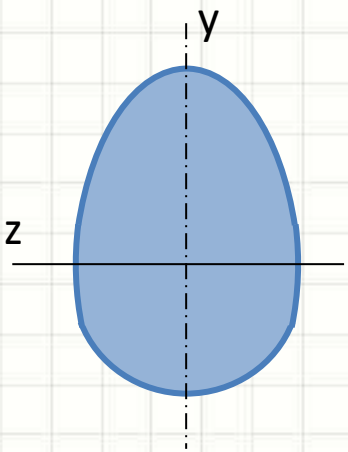
$$\sigma_{máx} = \frac{M.c}{I}$$

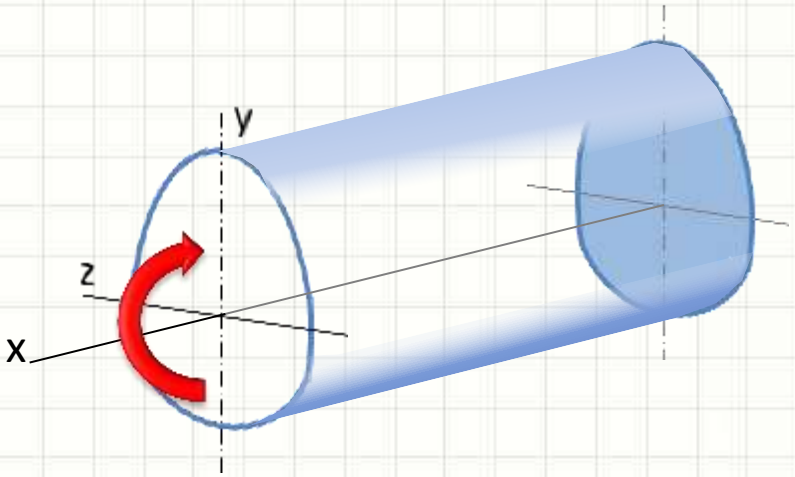
$$w = \frac{I_z}{c}$$

$$\sigma_{máx} = \frac{M}{w}$$

Deformação na Flexão

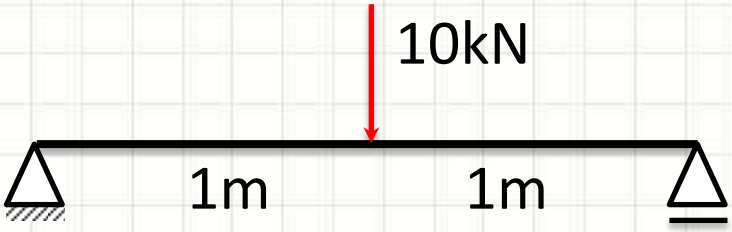
- Material Homogêneo e Alta Deformabilidade
- Seção transversal simétrica a um eixo
- Momento aplicado em torno de linha central perpendicular a esse eixo

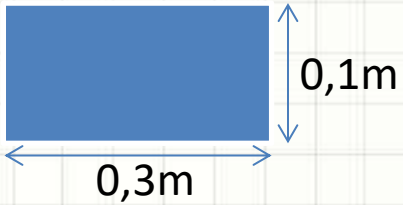




Exemplo

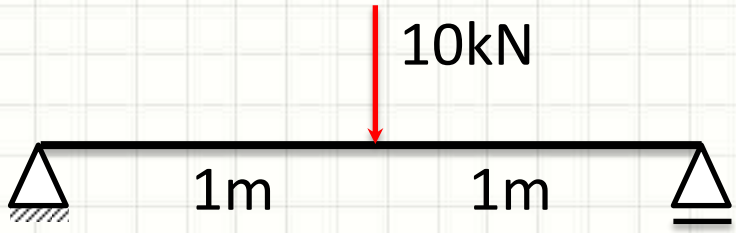
- Calcule a tensão de tração máxima:

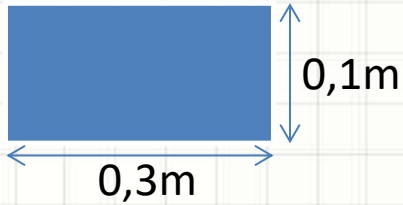


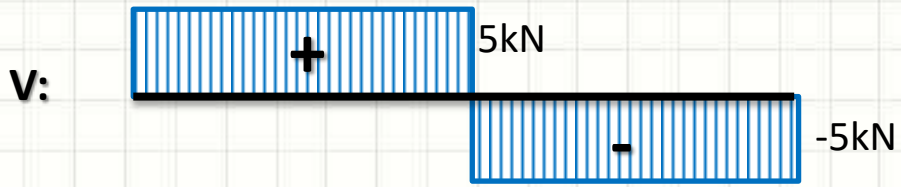


Exemplo

- Calcule a tensão de tração máxima:

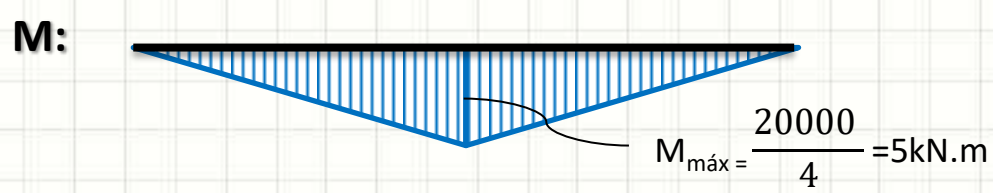






$$I = \frac{b.h^3}{12} = \frac{0,3.0,1^3}{12}$$

$$I = 0,000025m^4$$



$$\sigma_{máx} = \frac{M.c}{I} = \frac{5000.0,05}{0,000025} = 10MPa$$

Flexão Pura Reta

- Será que a teoria é limitada assim?
- Seção transversal “qualquer”
- Cargas combinadas
- Momento em qualquer direção

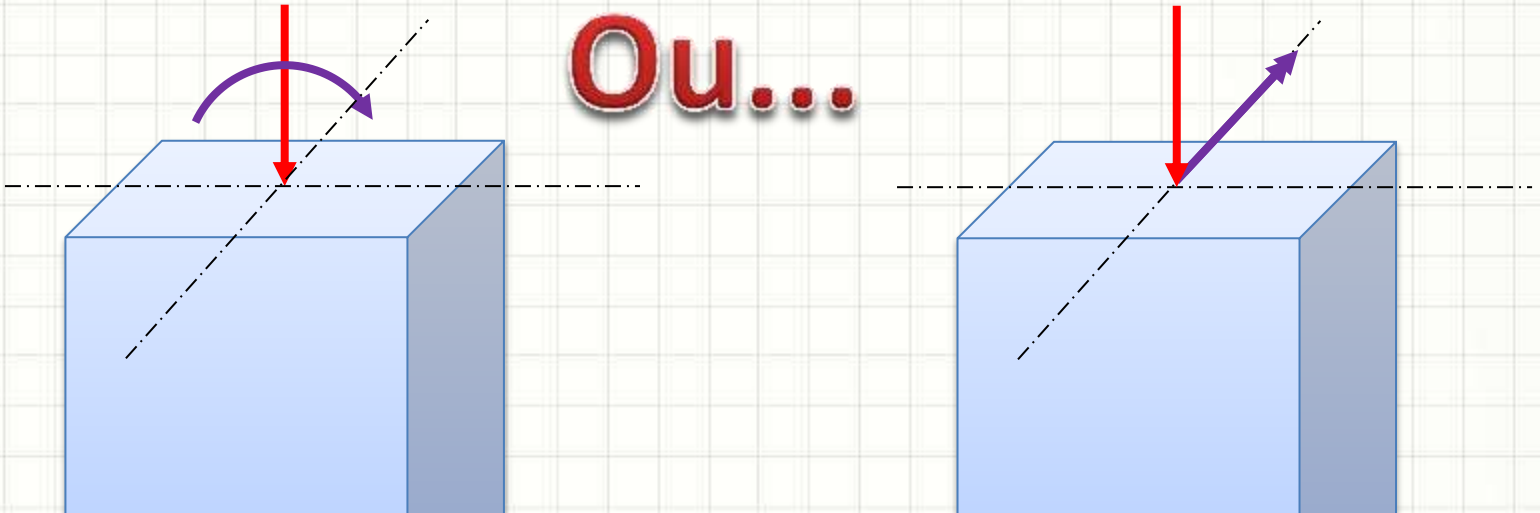
Flexo-Compressão

Flexão Oblíqua

FLEXO-COMPRESSÃO

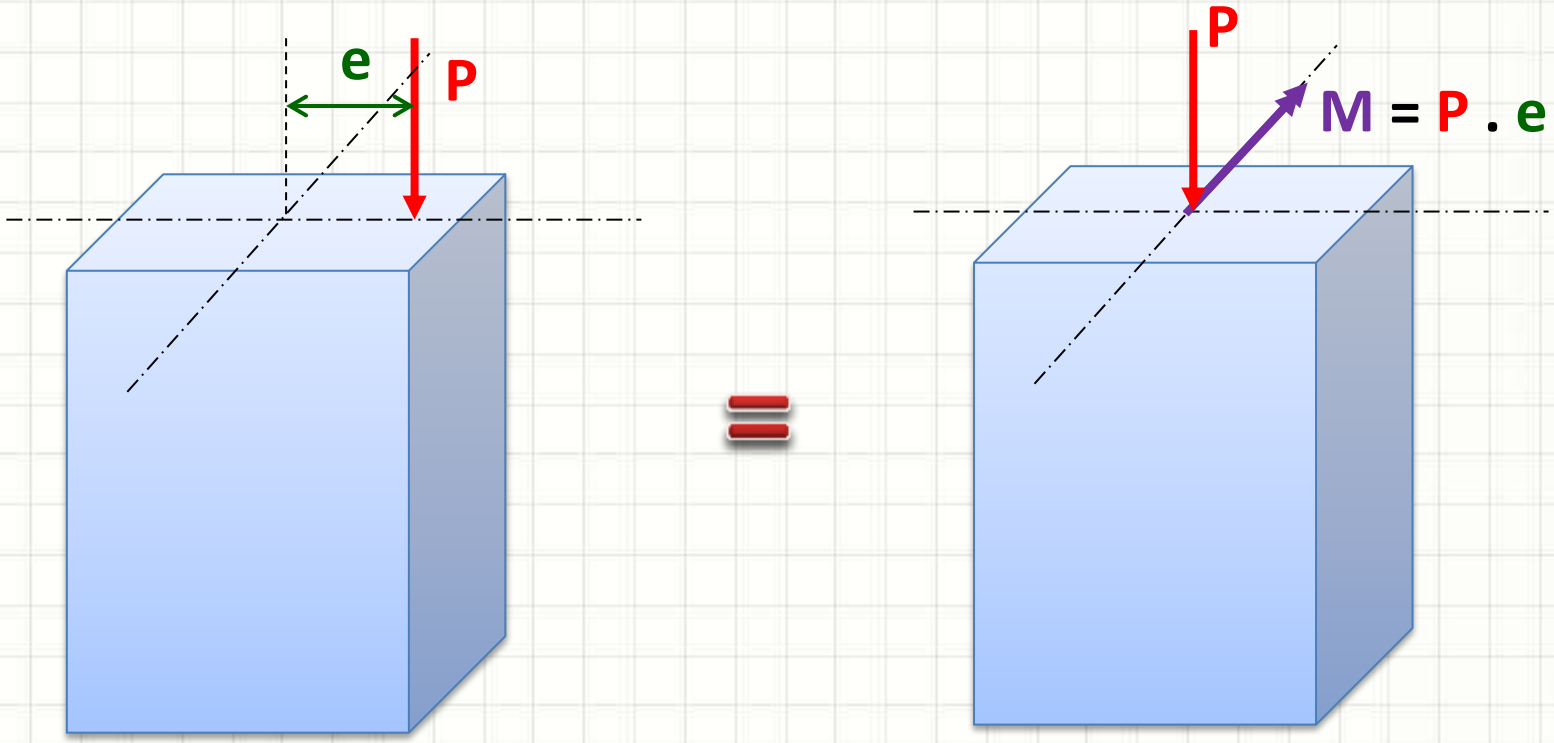
Flexo-Compressão

- Quando há flexão e compressão simultâneas
  - Comum em pilares/colunas
- Tratamento: princípio da superposição
  - Relação linear entre tensão e deformação
  - Geometria: não varia significativamente



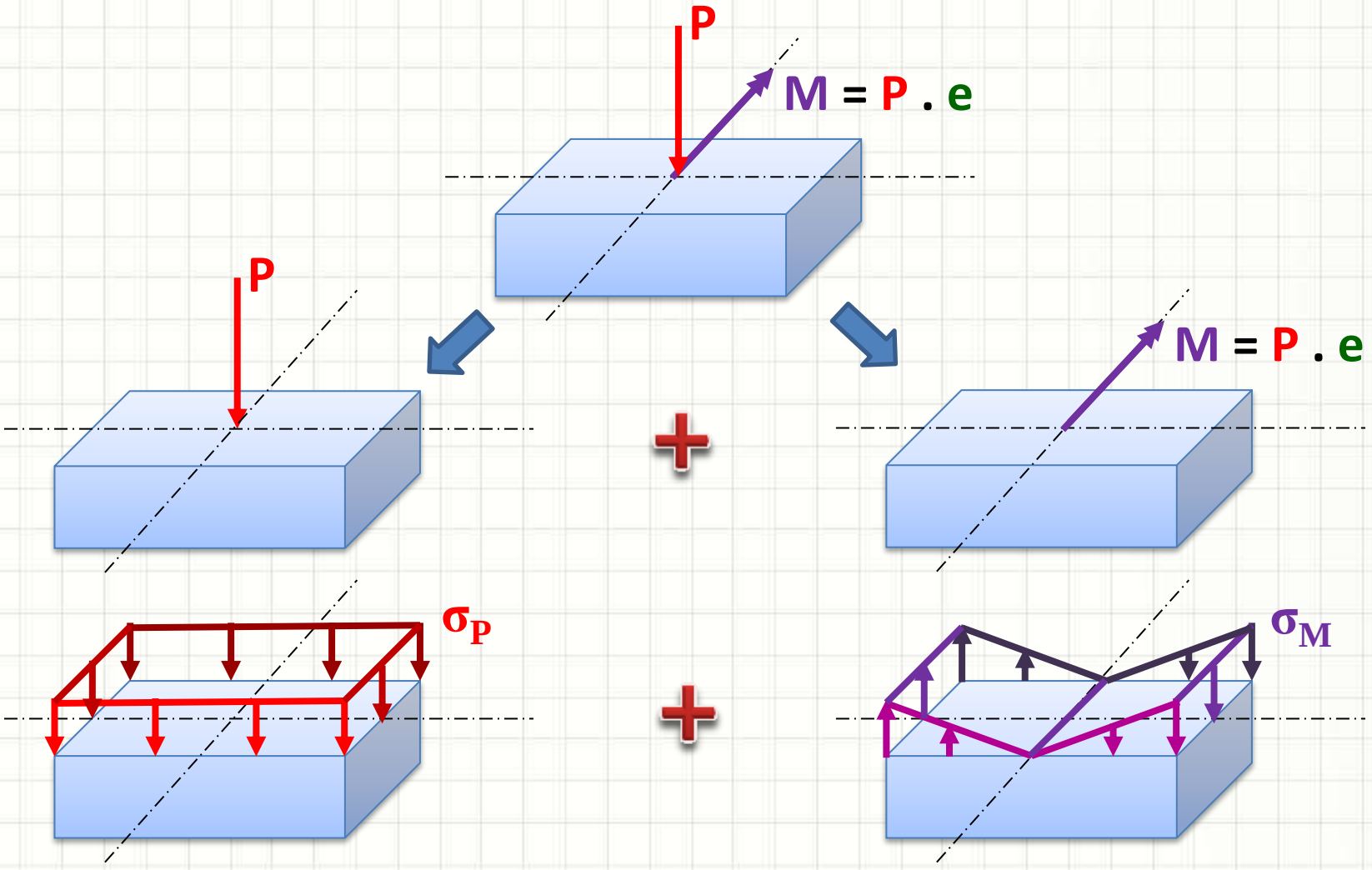
Flexo-Compressão

- Carga de compressão excêntrica
  - Pode ser de tratada como flexo-comrpessão



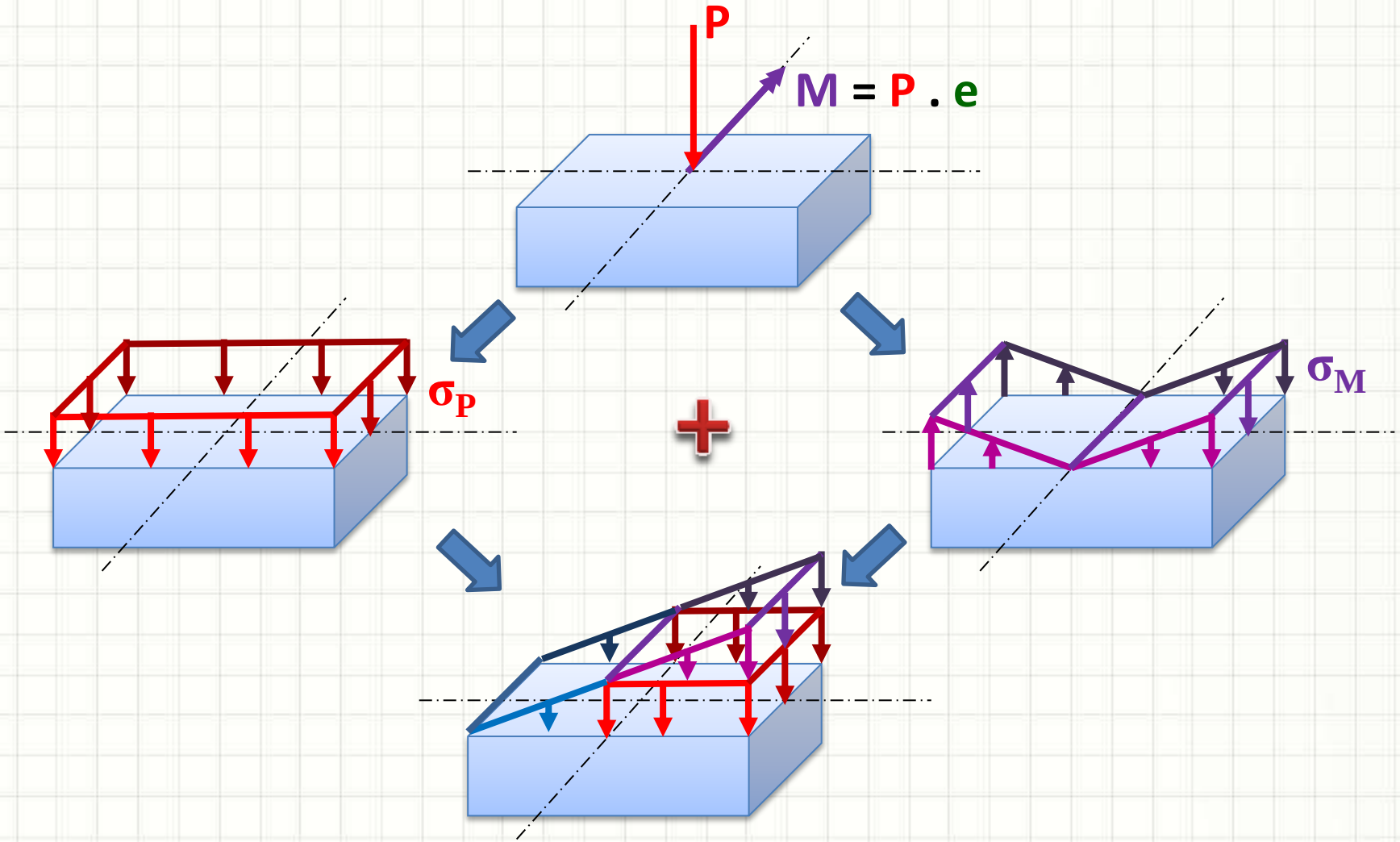
Flexo-Compressão

- Resolver por Superposição

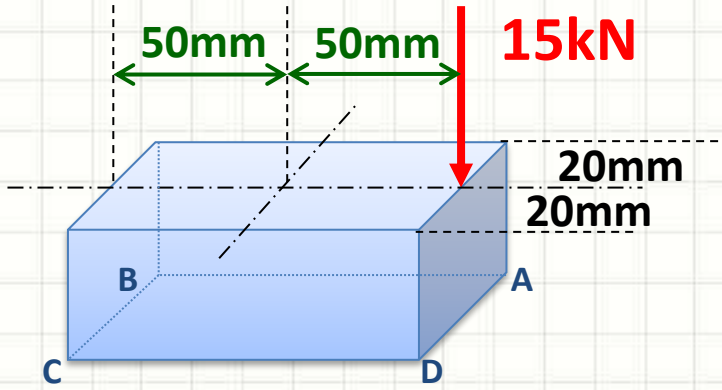


Flexo-Compressão

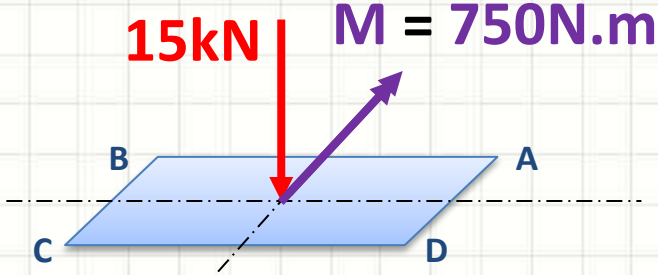
- Resolver por Superposição



Exemplo

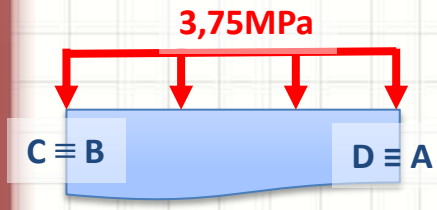


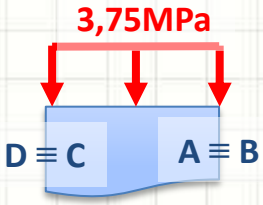
I) Cargas Atuantes



II) Força Normal

$$\sigma = \frac{F}{A} = \frac{15000}{0,1.0,04} = 3,75MPa$$

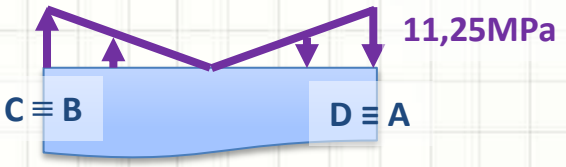




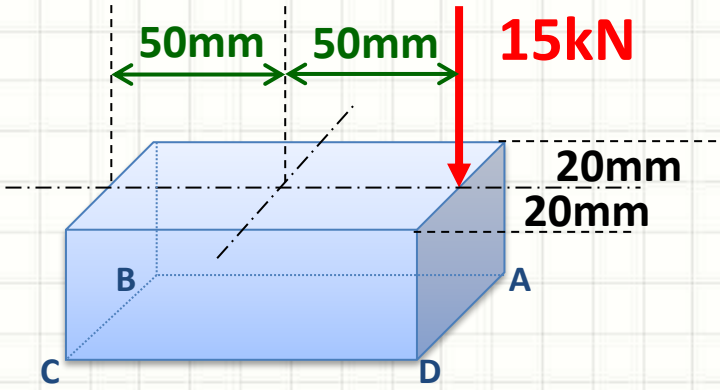
III) Momento Fletor

$$I = \frac{b.h^3}{12} = \frac{0,04.0,1^3}{12} = 3,33.10^{-6}\ m^4$$

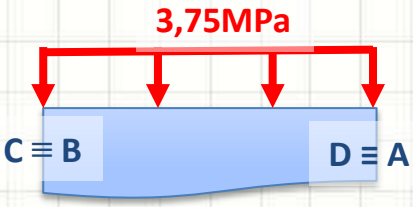
$$\sigma = \frac{M.y}{I} = \frac{750.0,05}{3,33.10^{-6}} \cong 11,25MPa$$

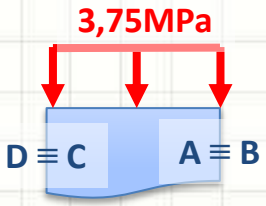


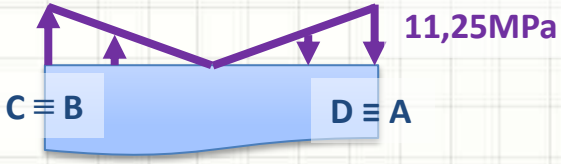
Exemplo



IV) Superposição

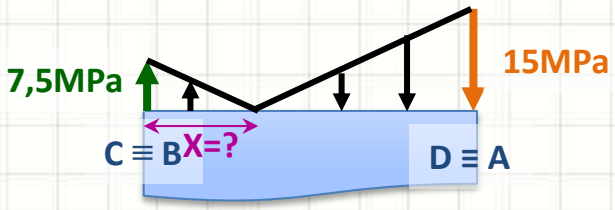






Em C/B: $\sigma = -3{,}75 + 11{,}25 = 7{,}5\text{MPa}$

Em D/A: $\sigma = -3{,}75 - 11{,}25 = -15\text{MPa}$



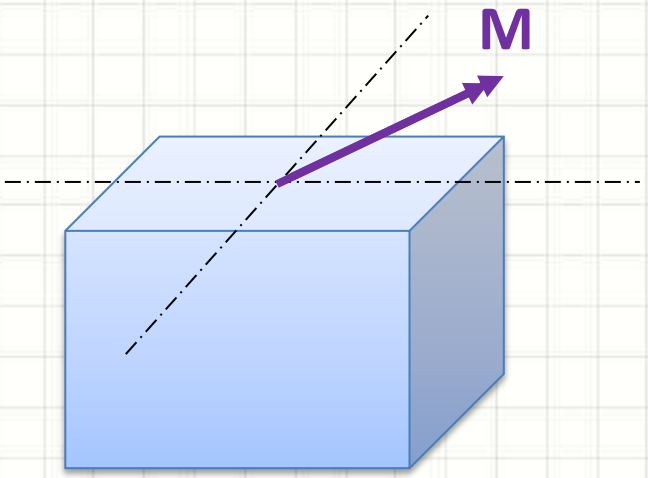
$$\frac{|\sigma_{comp}| + |\sigma_{tração}|}{largura} = \frac{\sigma}{x}$$

$$\frac{15.10^6 + 7{,}5.10^6}{0{,}1} = \frac{7{,}5.10^6}{x}$$

$$\boxed{x = 0{,}033m}$$

Dúvida Cruel

- Podemos aplicar $\sigma_{máx} = \frac{M.c}{I}$ nesse caso?

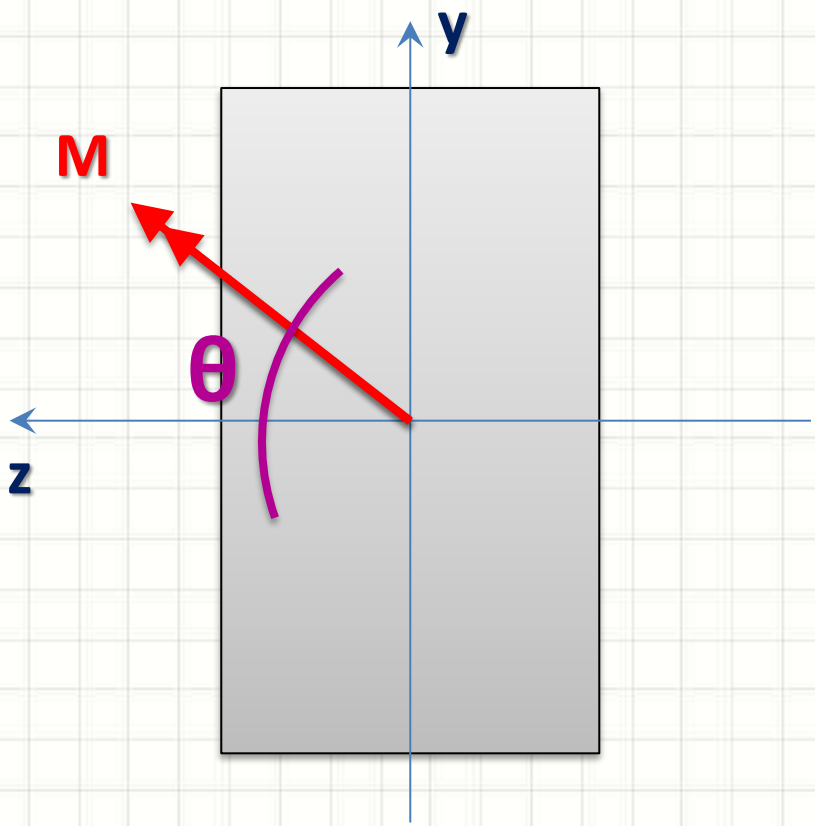


- Diretamente, não...
  - Premissa: ser ao redor de eixo perp. ao de simetria
- #Comofaz?

Momentos Oblíquos e a Fórmula da Flexão Generalizada

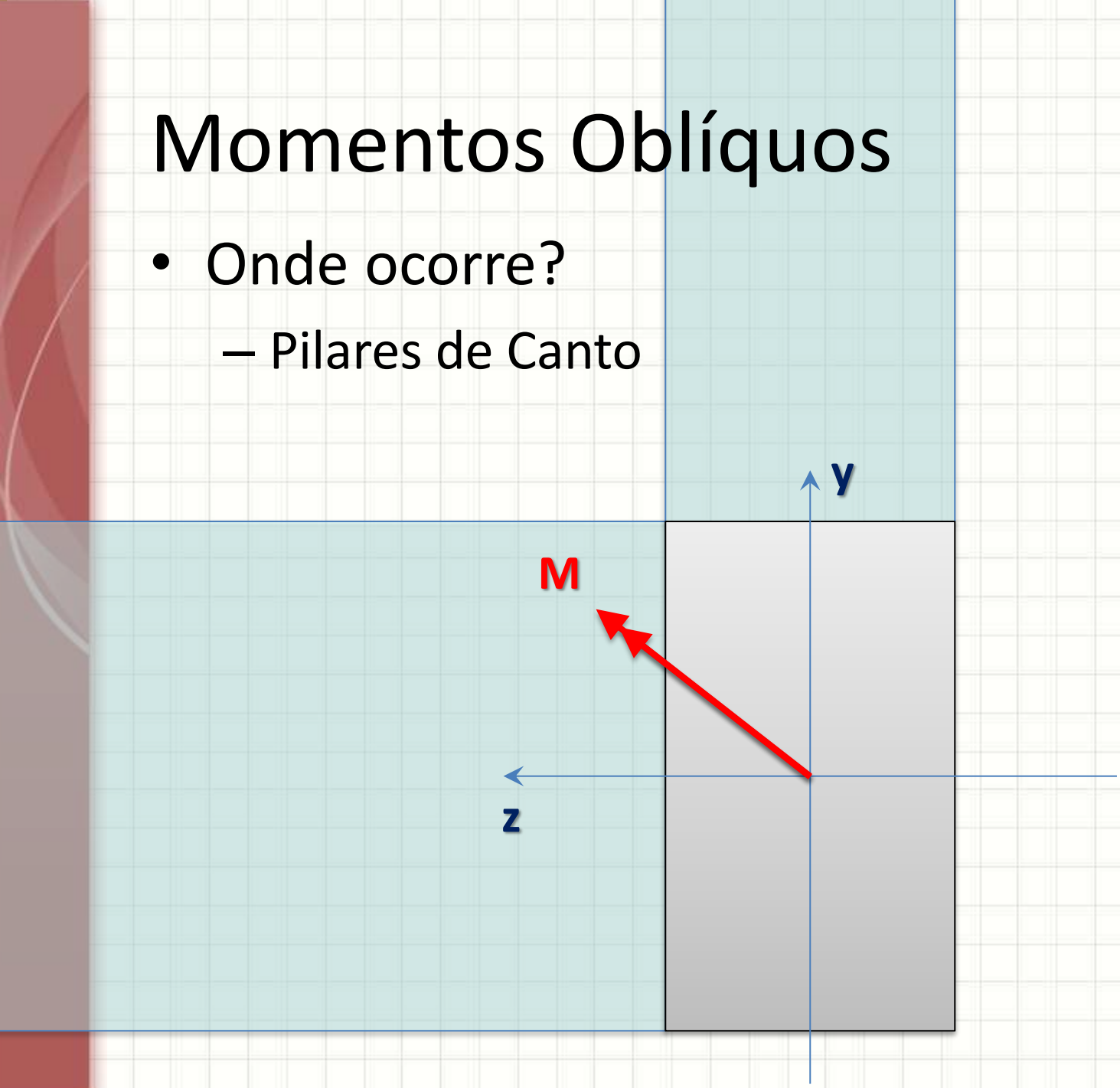
Momentos Oblíquos

- Momento Oblíquo:
  - Não é em torno de eixo perp. ao de simetria



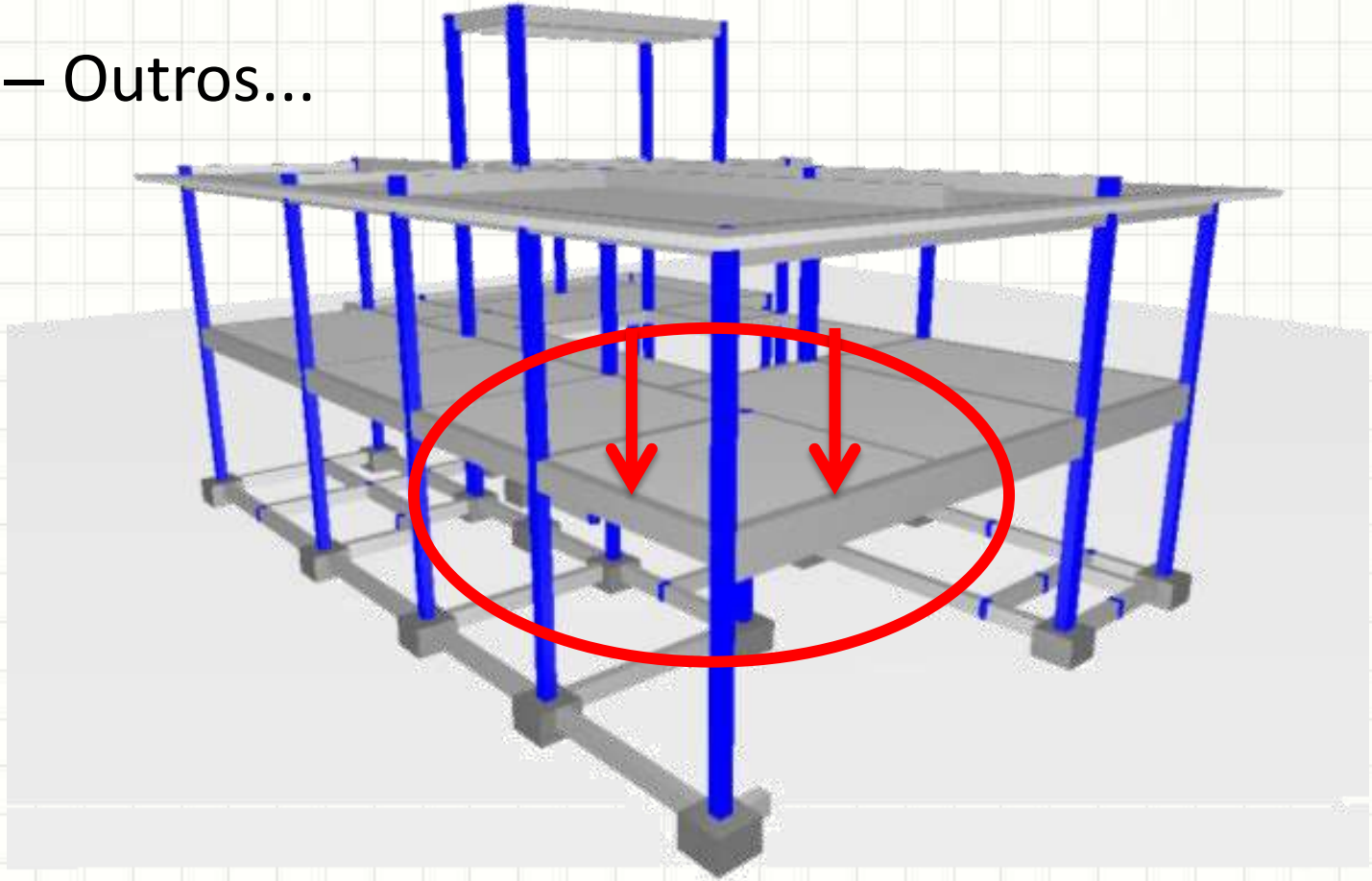
Momentos Oblíquos

- Onde ocorre?
  - Pilares de Canto



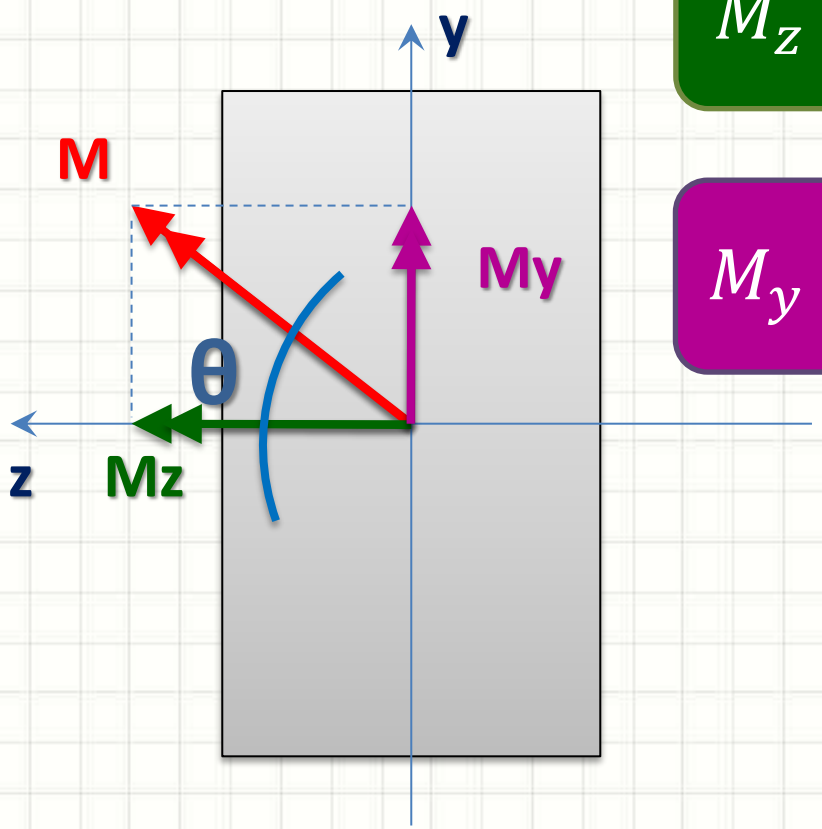
Momentos Oblíquos

- Onde ocorre?
  - Pilares de Canto
  - Outros...



Momentos Oblíquos

- Não são em torno de eixo perp. ao de simetria
  - Mas podemos decompô-los

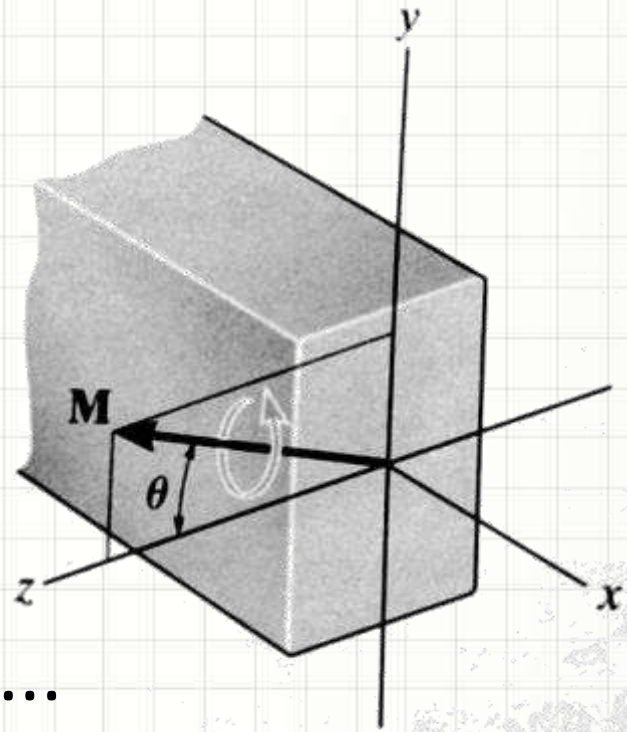


$$M_z = M.\cos\theta_z$$

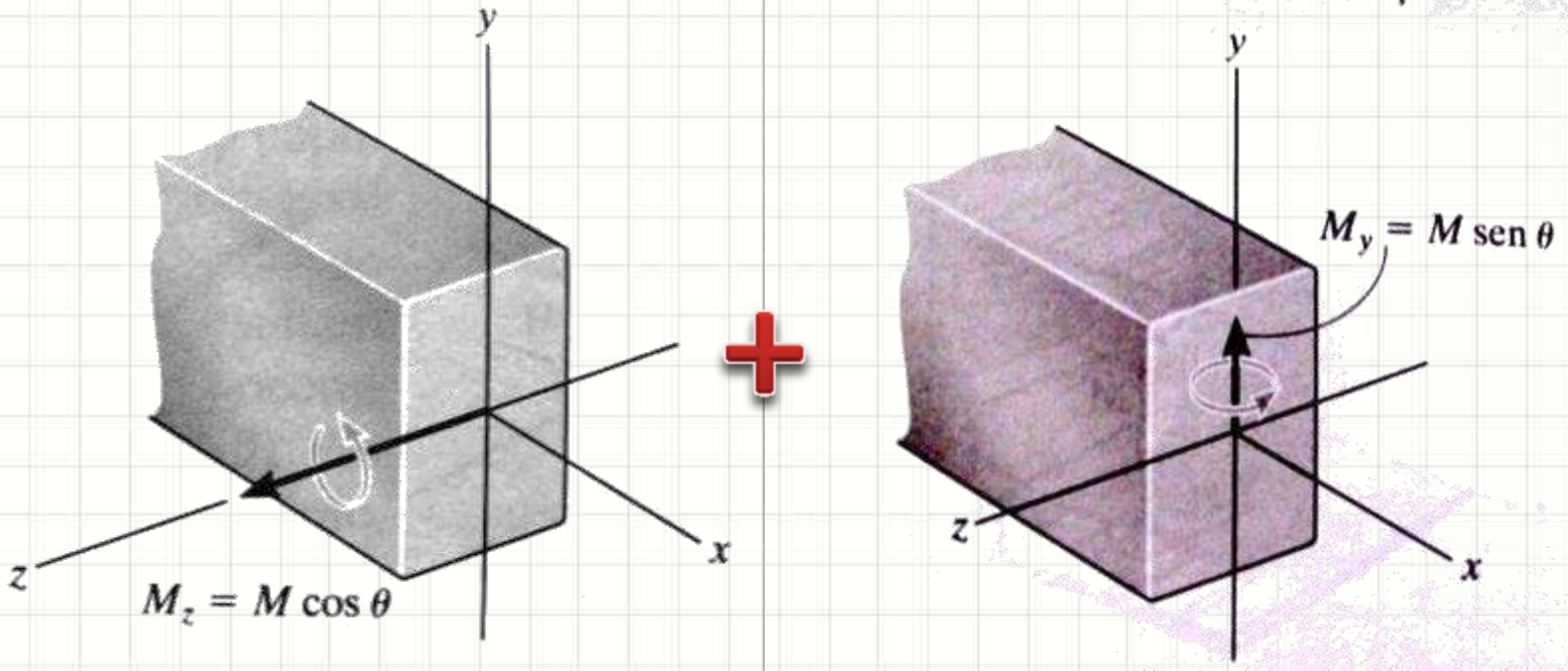
$$M_y = M.\operatorname{sen}\theta_z$$

Momentos Oblíquos

- Visão em Perspectiva

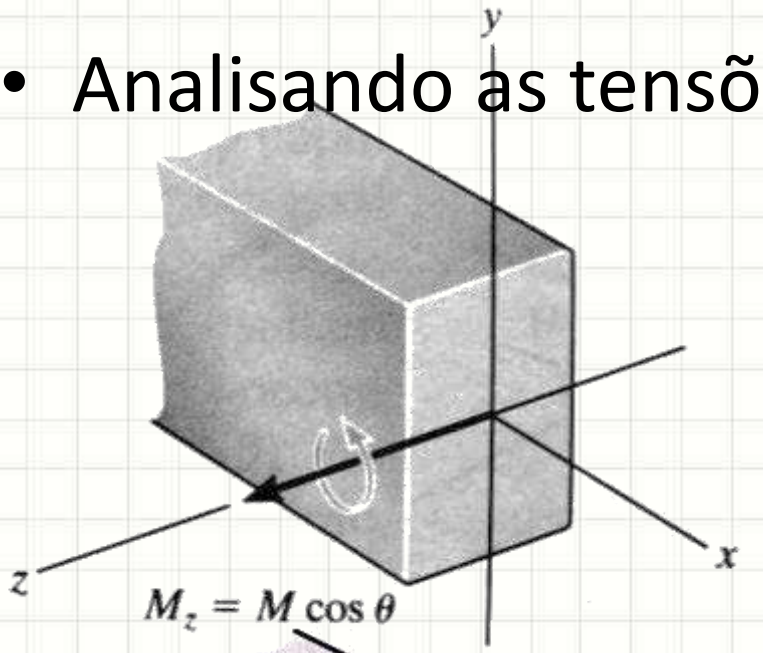


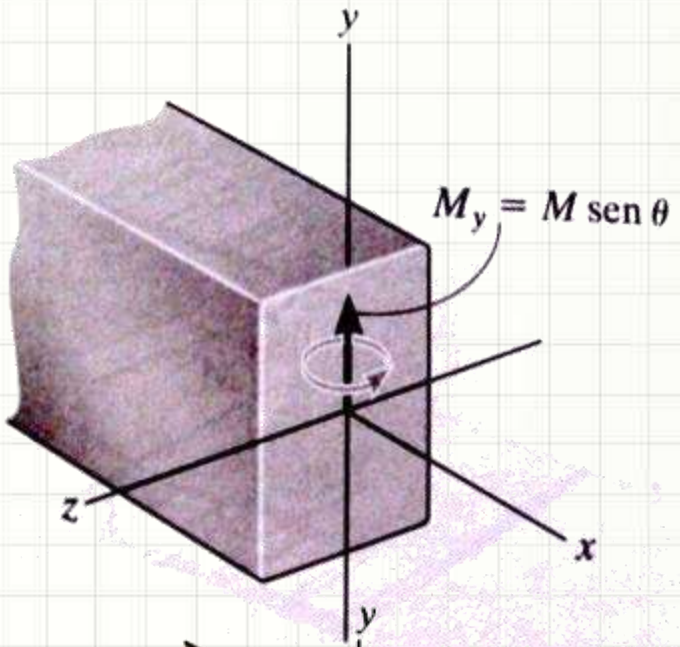
- Por superposição de efeitos...

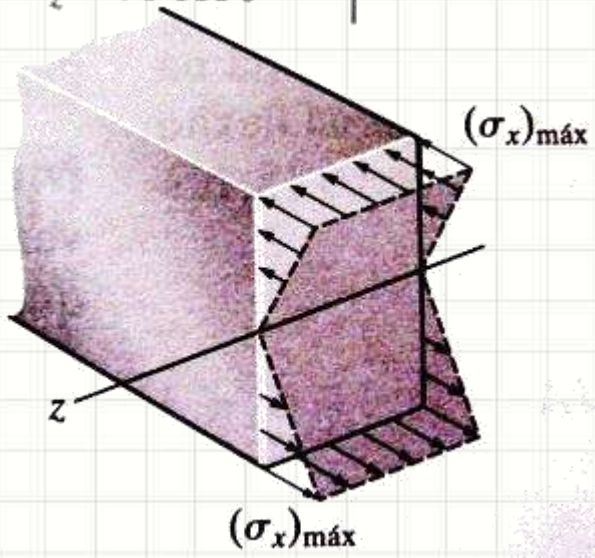


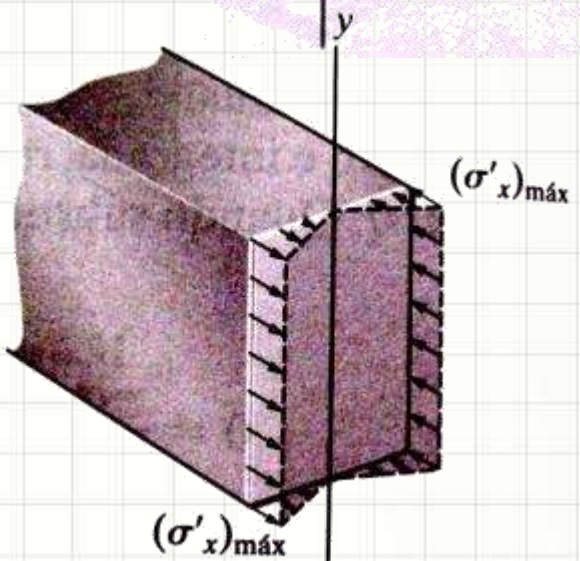
Momentos Oblíquos

- Analisando as tensões



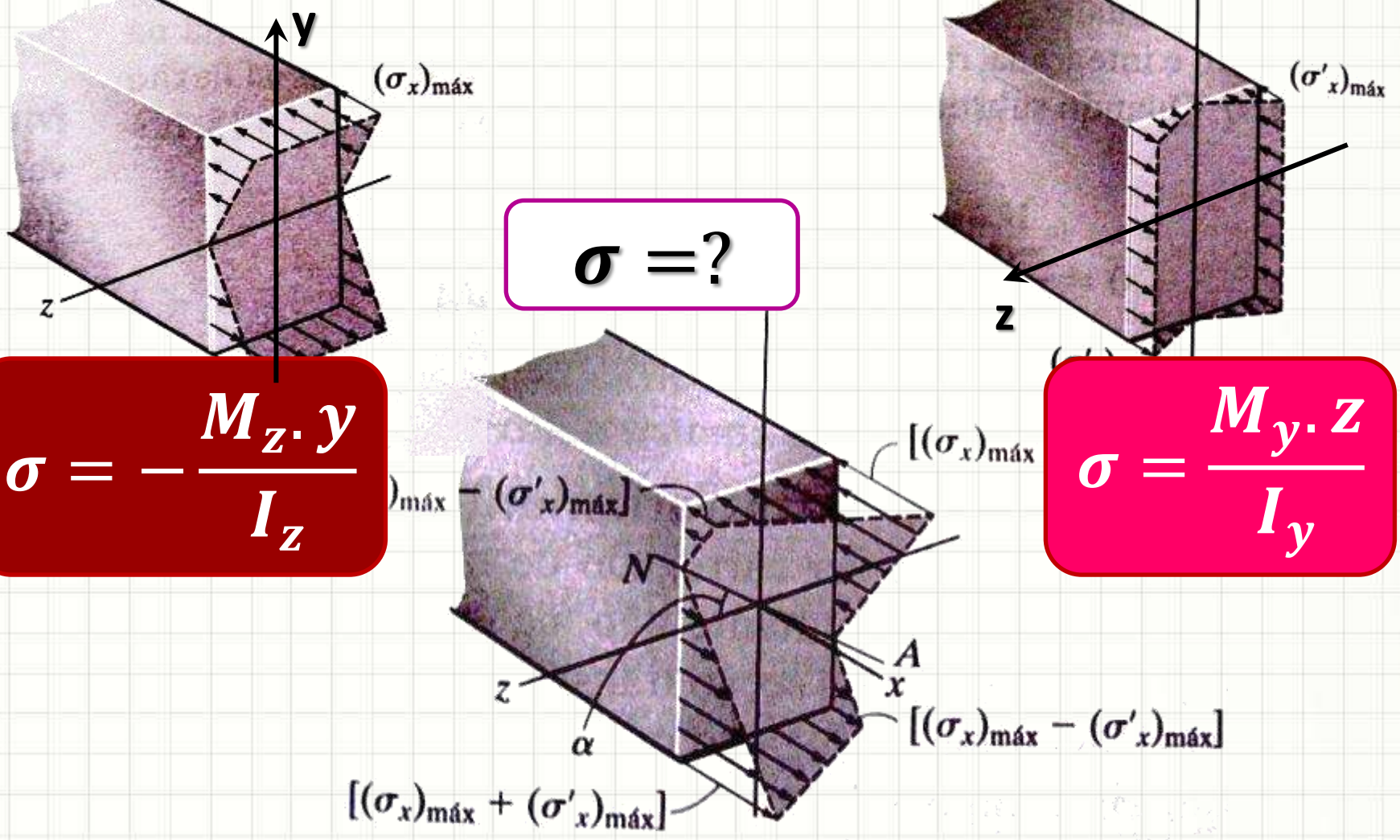






Momentos Oblíquos

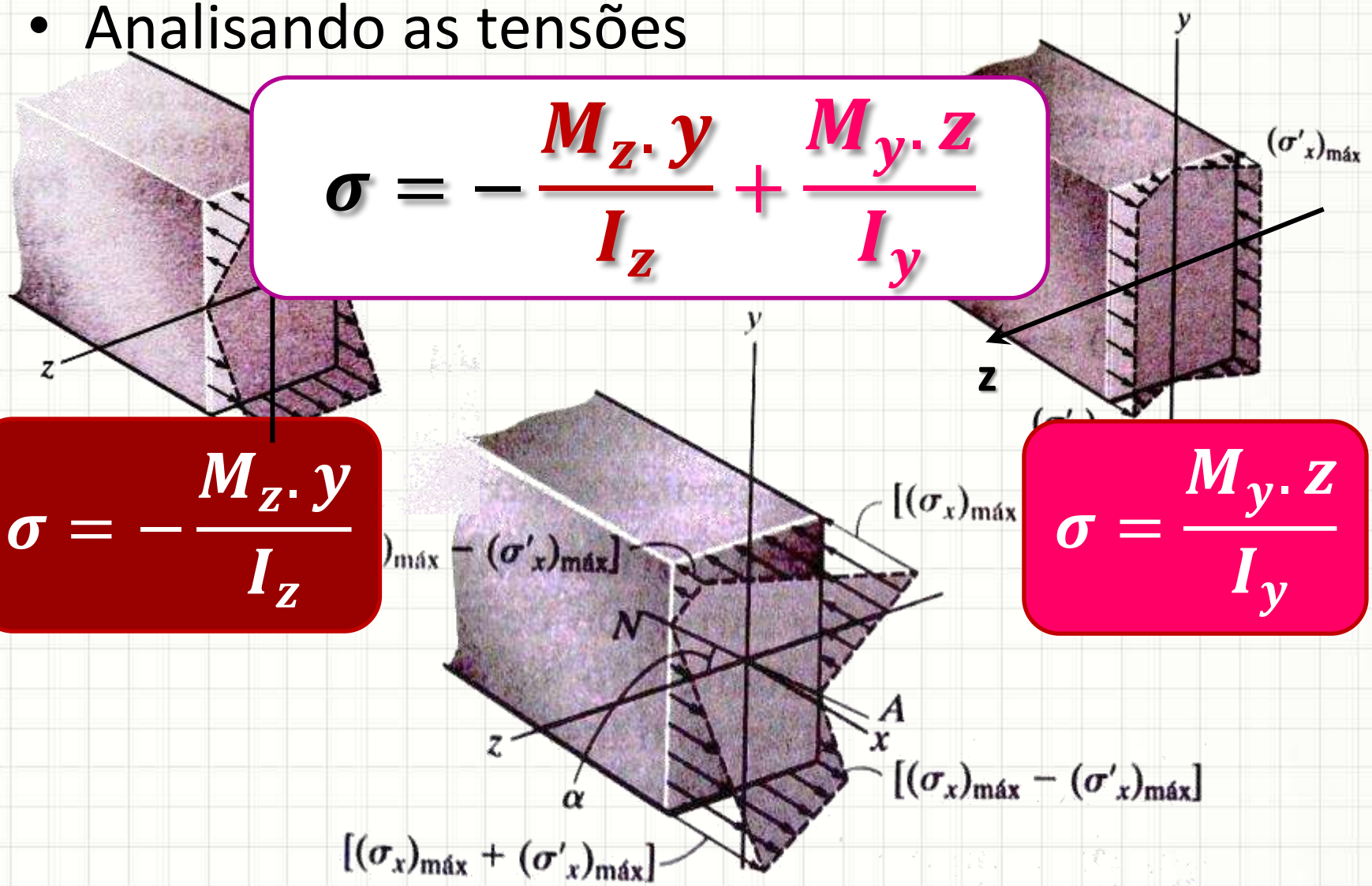
- Analisando as tensões



Momentos Oblíquos

- Analisando as tensões

$$\sigma = -\frac{M_z . y}{I_z} + \frac{M_y . z}{I_y}$$

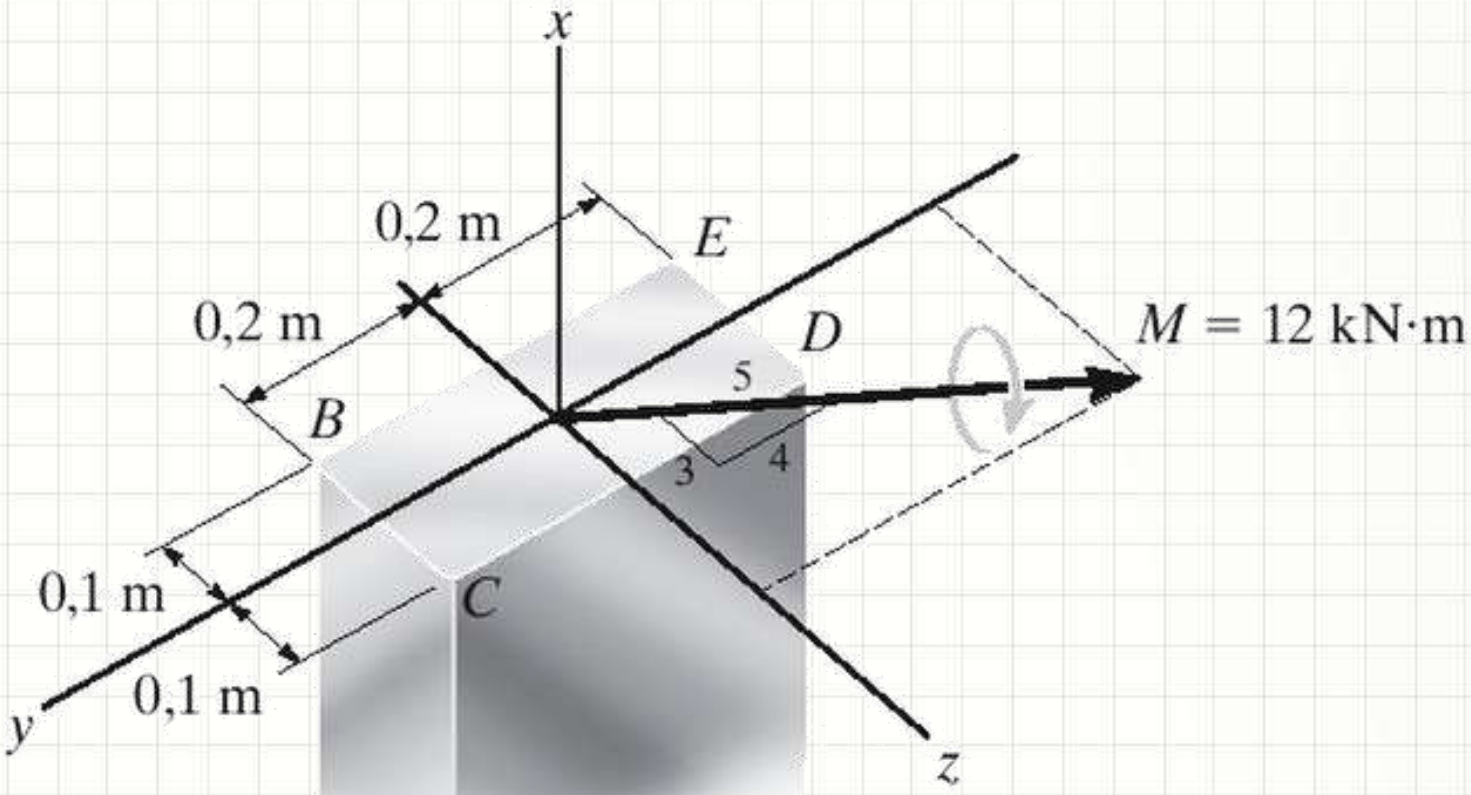


$$\sigma = -\frac{M_z . y}{I_z}$$

$$\sigma = \frac{M_y . z}{I_y}$$

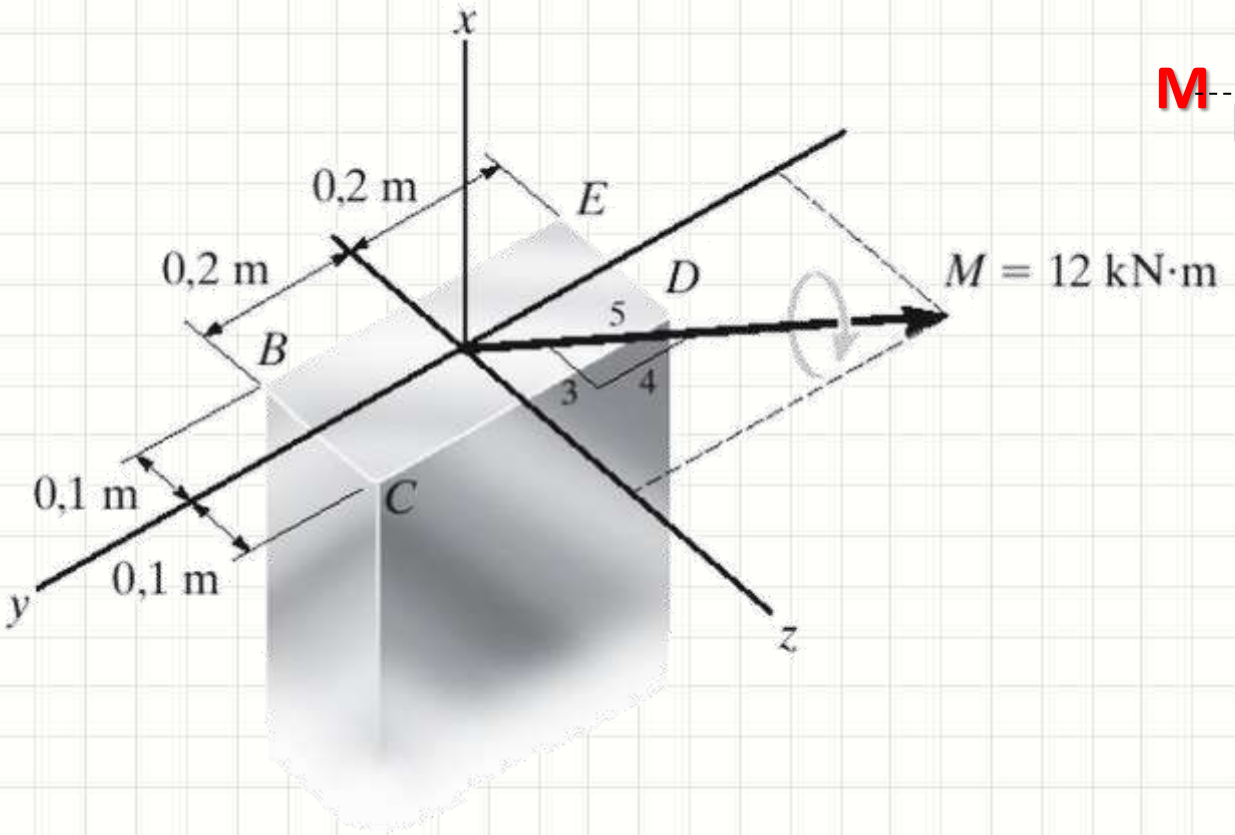
Exemplo

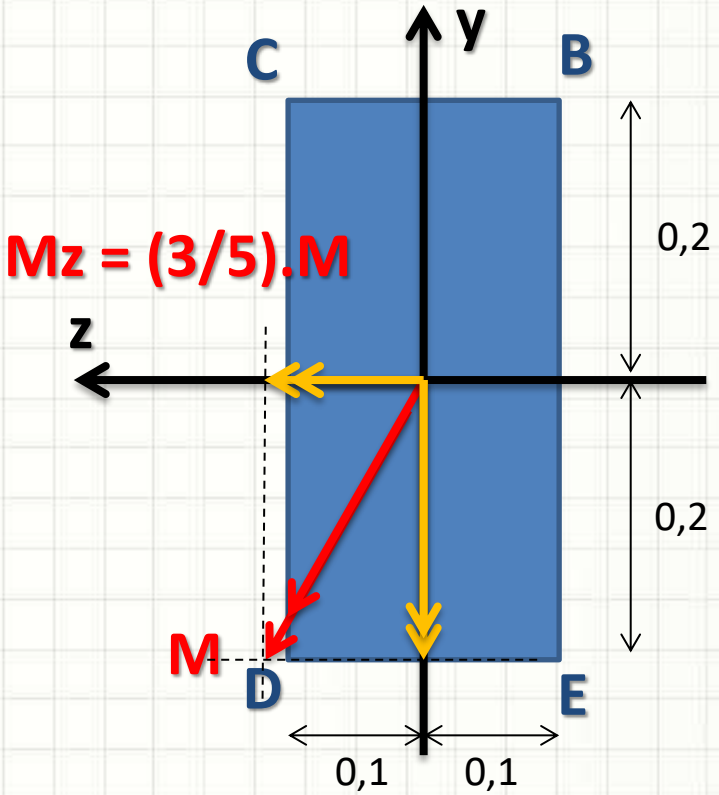
- Considerando M=12kN.m, indique a tensão em cada canto da seção transversal



Exemplo

- M=12kN.m, σ_B a σ_E



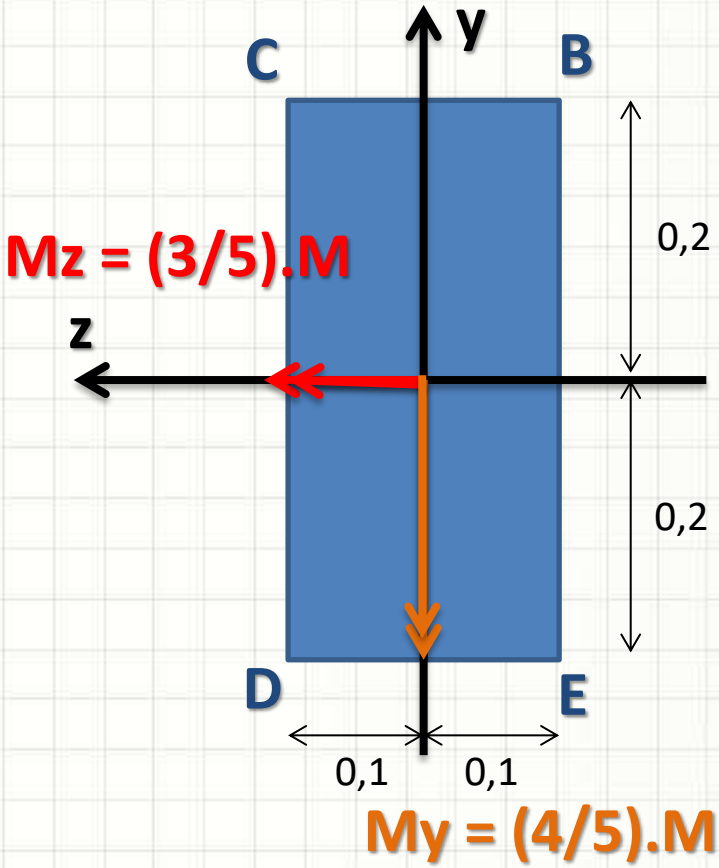


Exemplo

- M=12kN.m, σ_B a σ_E

$$M_z = \frac{3.M}{5} = \frac{3.12000}{5} = 7{,}2kN.m$$

$$M_y = \frac{4.M}{5} = \frac{4.12000}{5} = 9{,}6kN.m$$



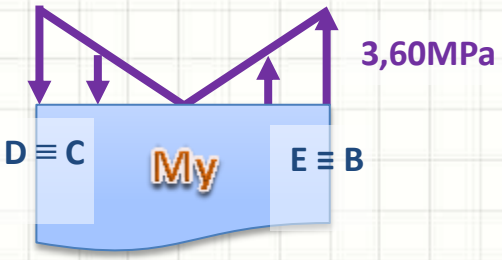
Exemplo

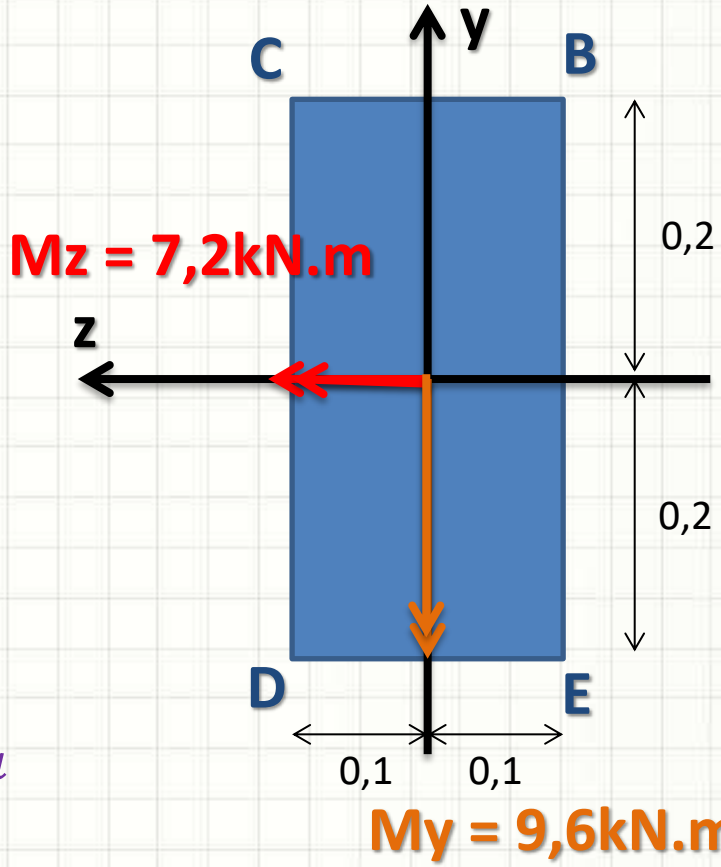
- M=12kN.m, σ_B a σ_E

I) Momento My

$$I_y = \frac{b.h^3}{12} = \frac{0{,}4.0{,}2^3}{12} = 0{,}000266\ldots m^4$$

$$\sigma = \frac{M_y.z}{I_y} = \frac{9600.0{,}1}{2{,}66667.10^{-4}} \cong 3{,}60MPa$$





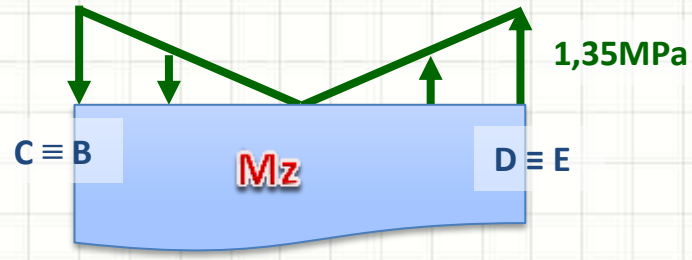
Exemplo

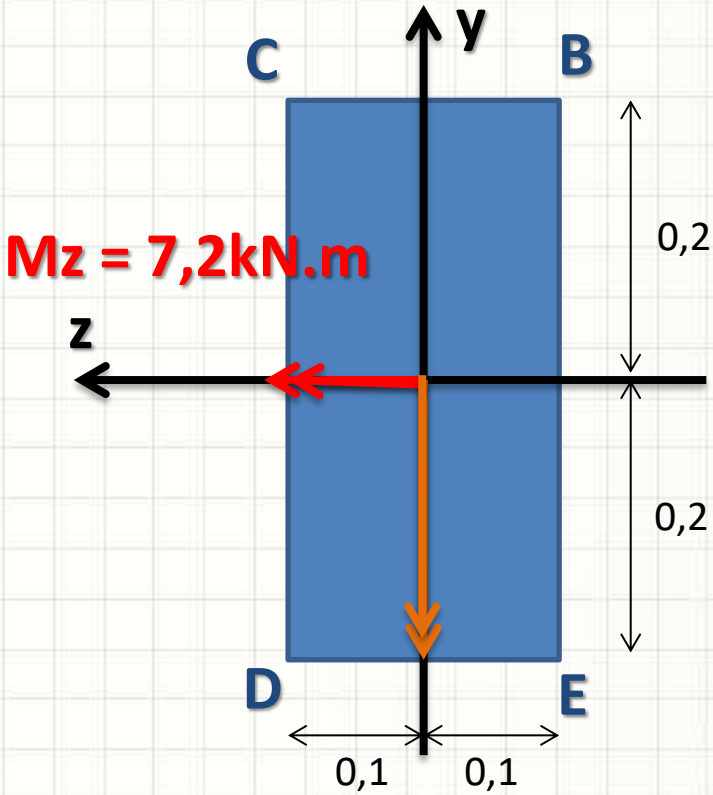
- M=12kN.m, σ_B a σ_E

II) Momento Mz

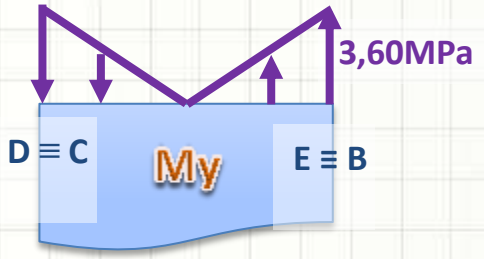
$$I_z = \frac{b.h^3}{12} = \frac{0{,}2.0{,}4^3}{12} = 0{,}0010666\ldots m^4$$

$$\sigma = \frac{M_z.y}{I_z} = \frac{7200.0{,}2}{1{,}067.10^{-3}} \cong 1{,}35MPa$$





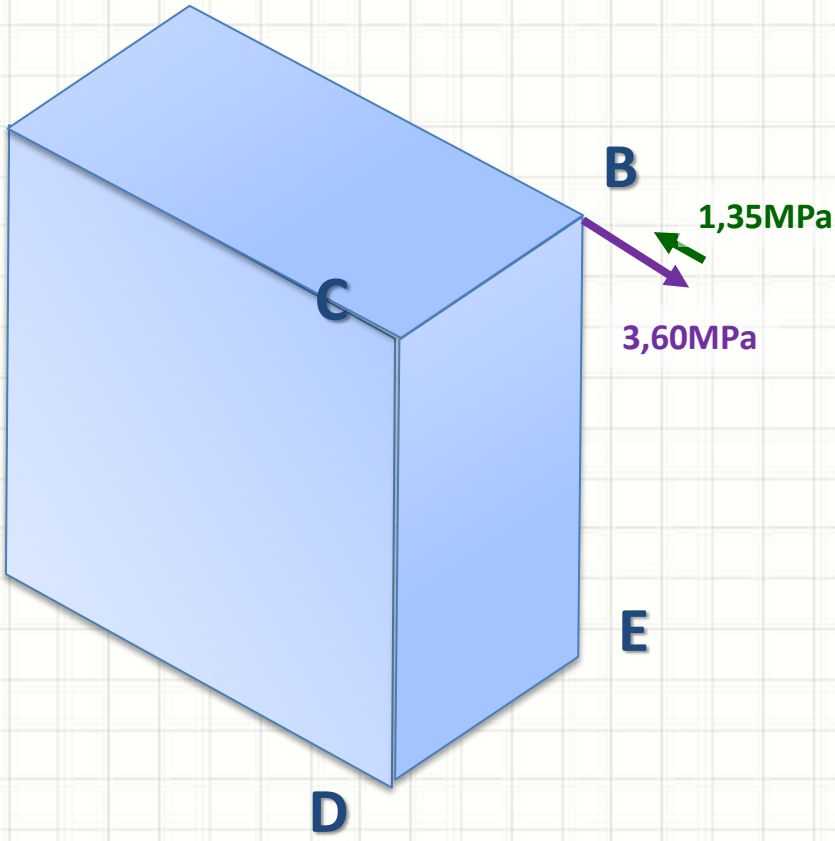
My = 9,6kN.m

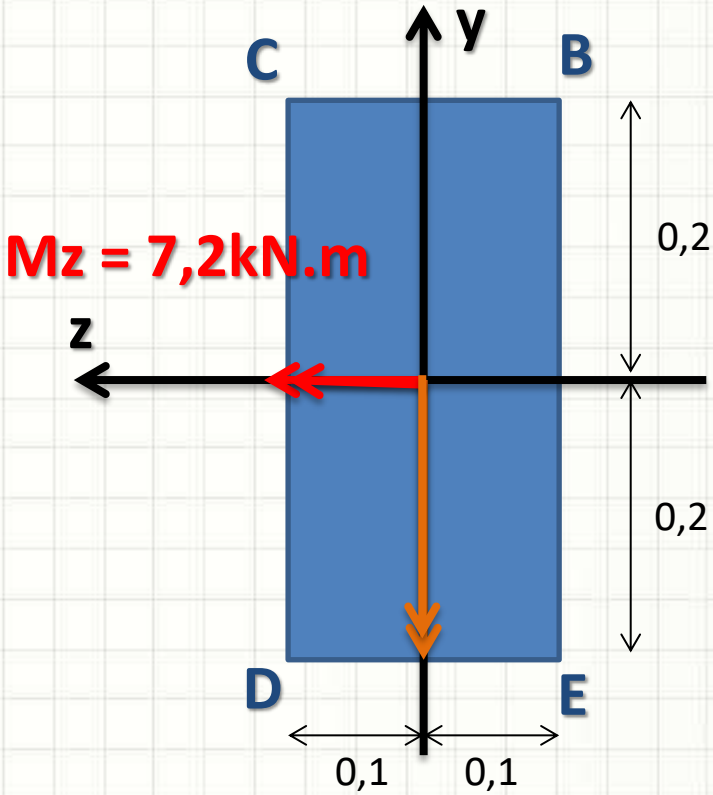


Exemplo

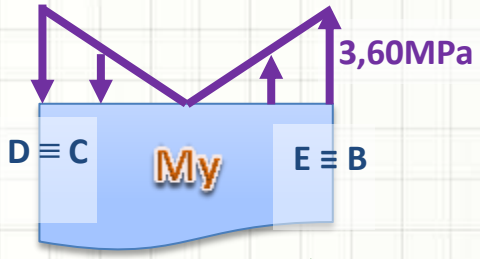
- M=12kN.m, σ_B a σ_E

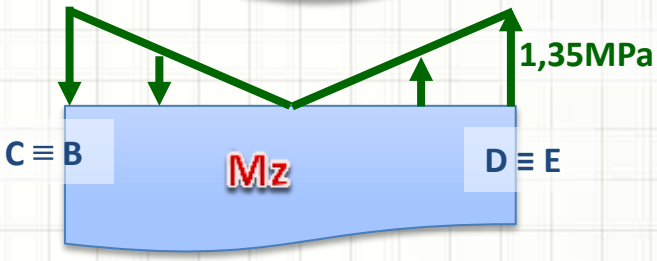
III) Sobreposição





My = 9,6kN.m

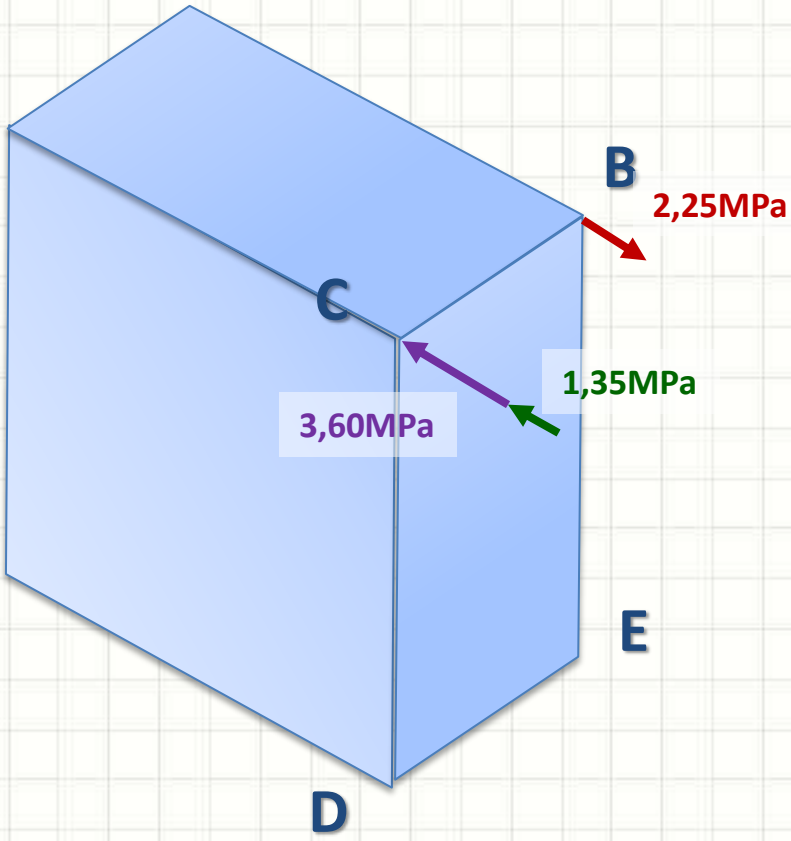


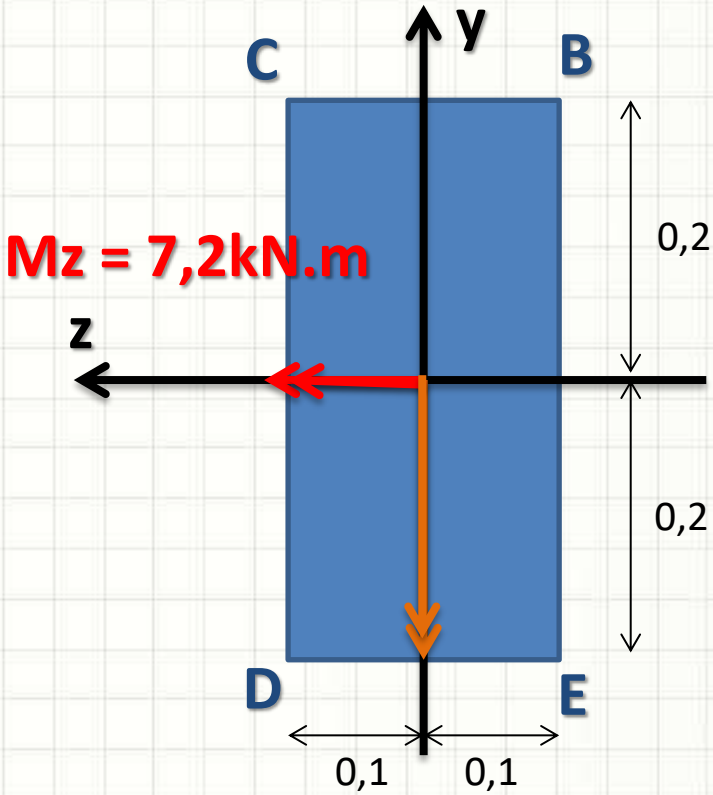


Exemplo

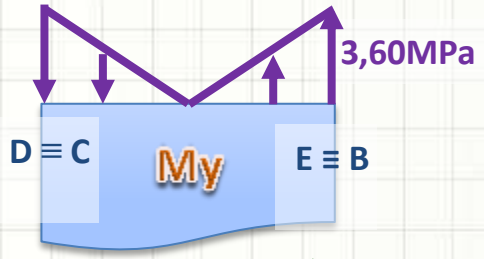
- M=12kN.m, σ_B a σ_E

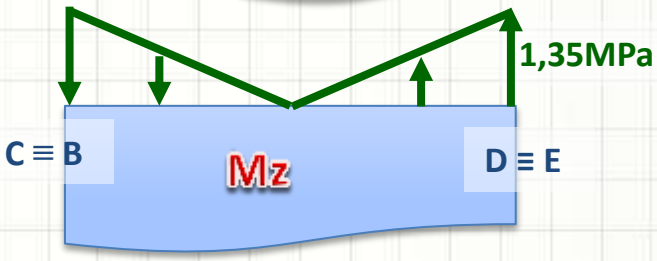
III) Sobreposição





My = 9,6kN.m

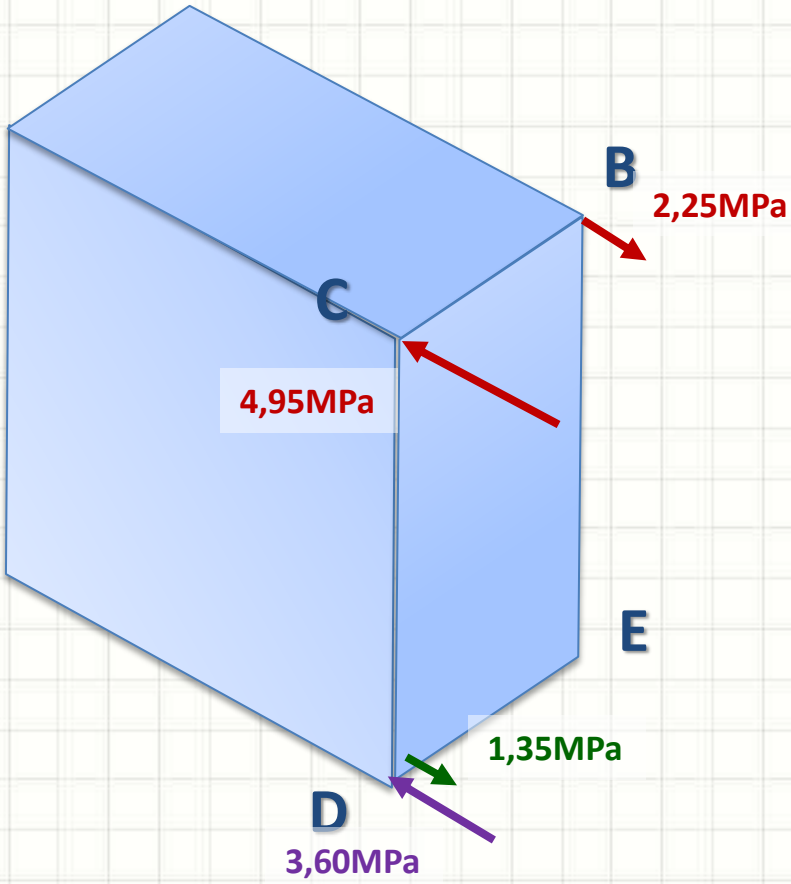


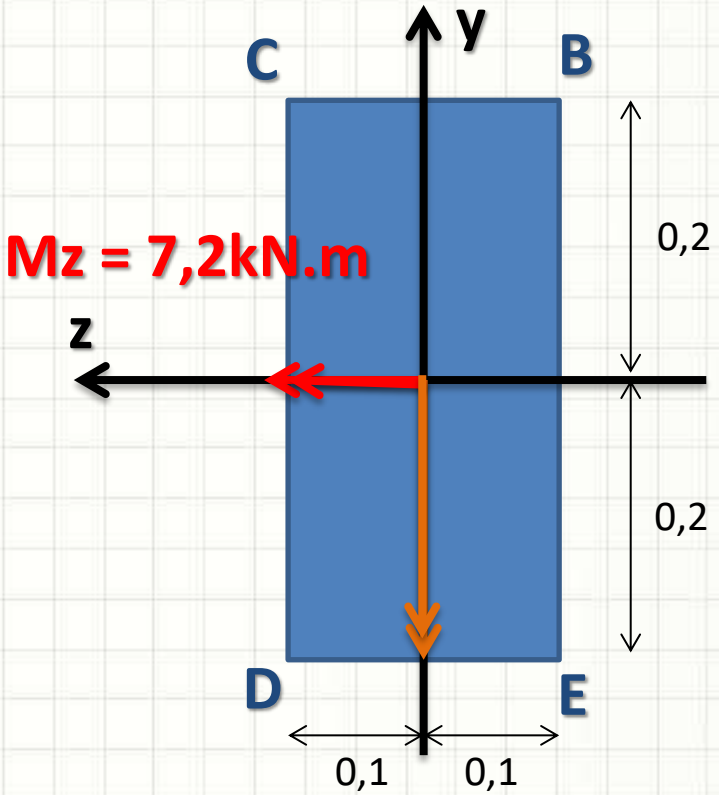


Exemplo

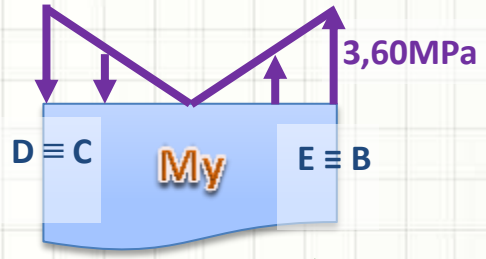
- M=12kN.m, σ_B a σ_E

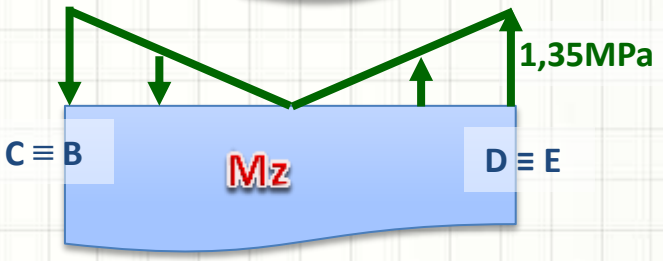
III) Sobreposição





My = 9,6kN.m

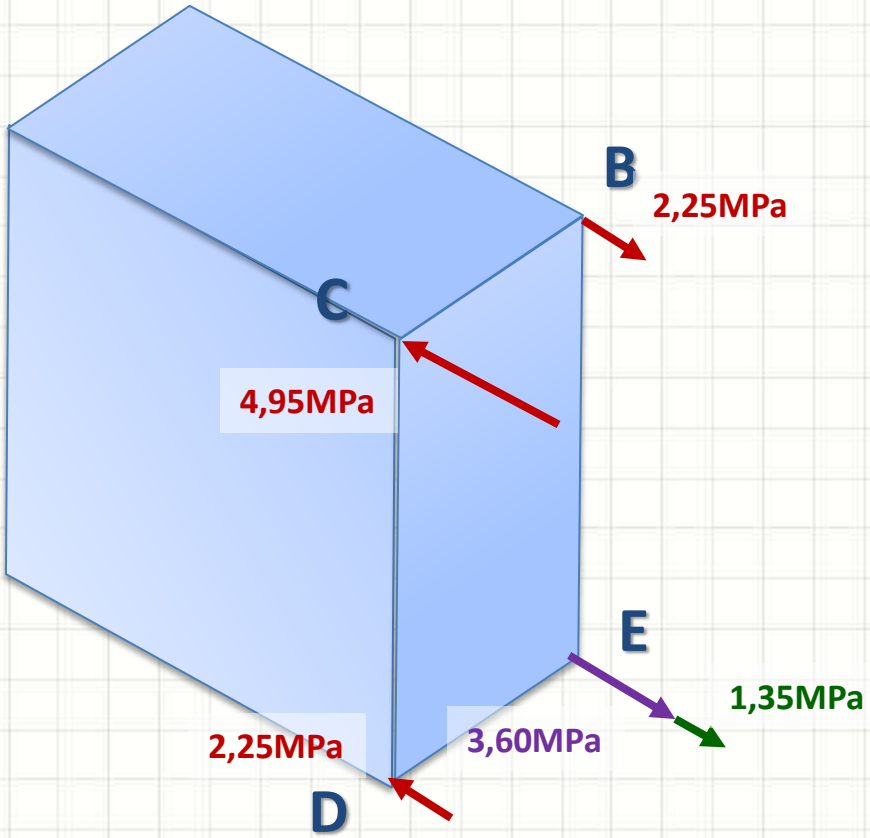


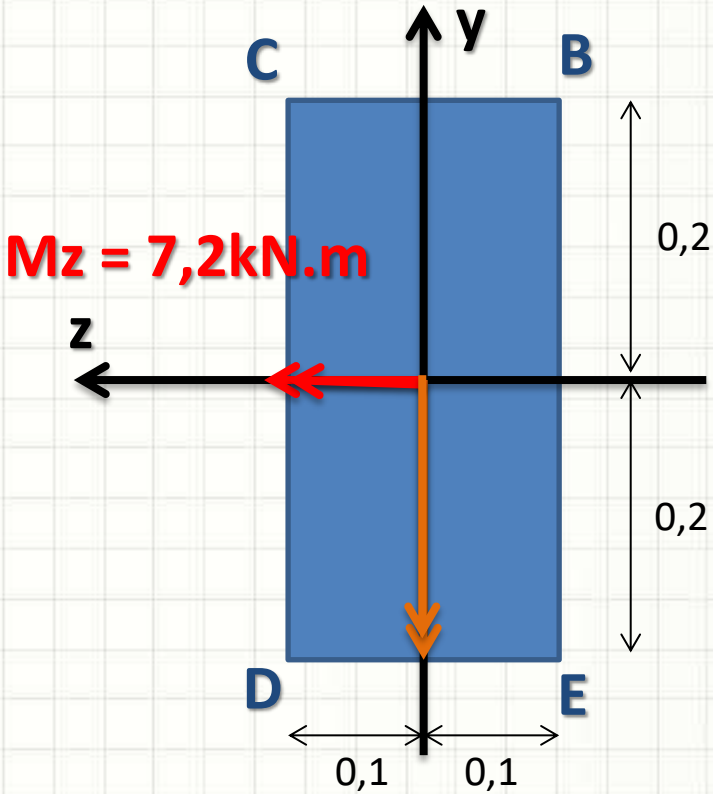


Exemplo

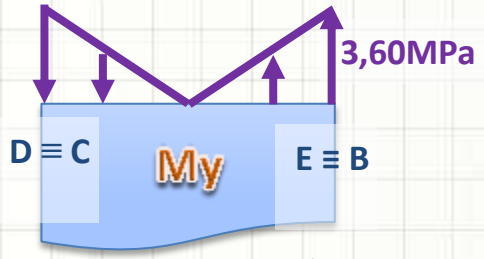
- M=12kN.m, σ_B a σ_E

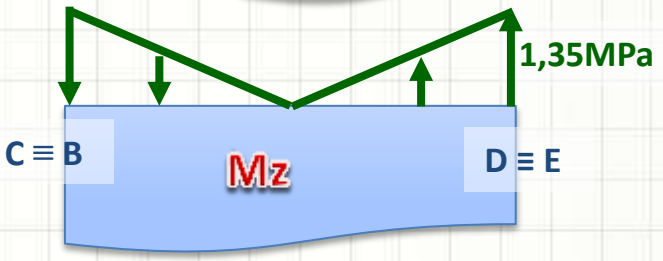
III) Sobreposição





My = 9,6kN.m





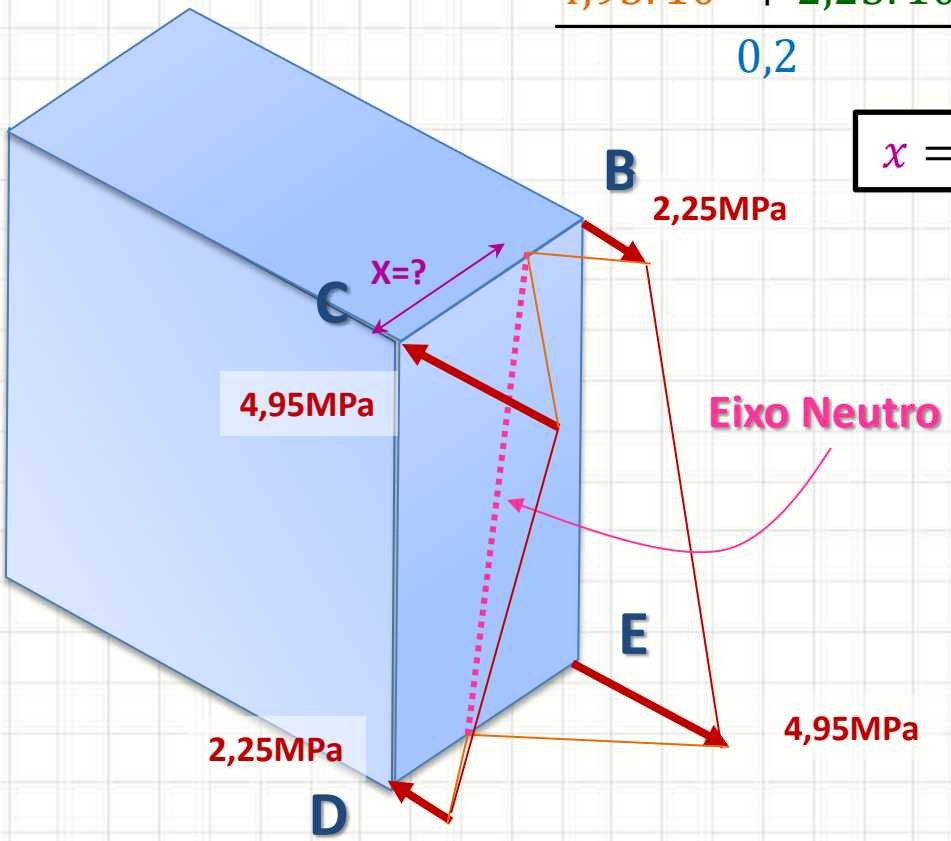
Exemplo

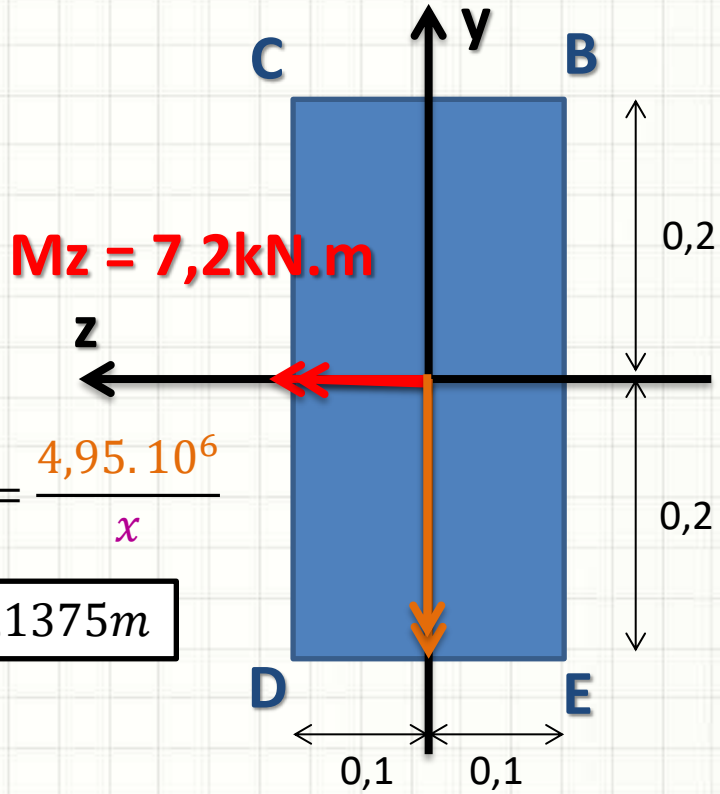
- M=12kN.m, σ_B a σ_E

III) Sobreposição

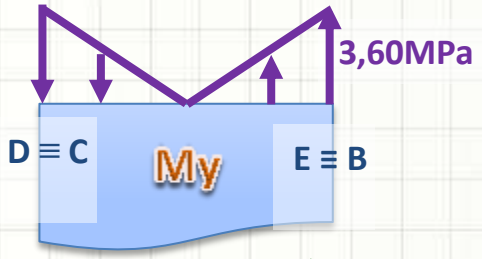
$$\frac{4{,}95.10^6 + 2{,}25.10^6}{0{,}2} = \frac{4{,}95.10^6}{x}$$

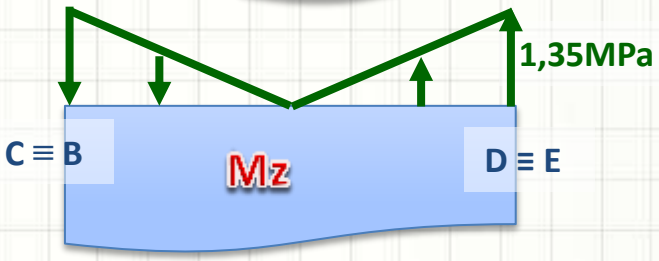
$$x = 0{,}1375m$$





My = 9,6kN.m

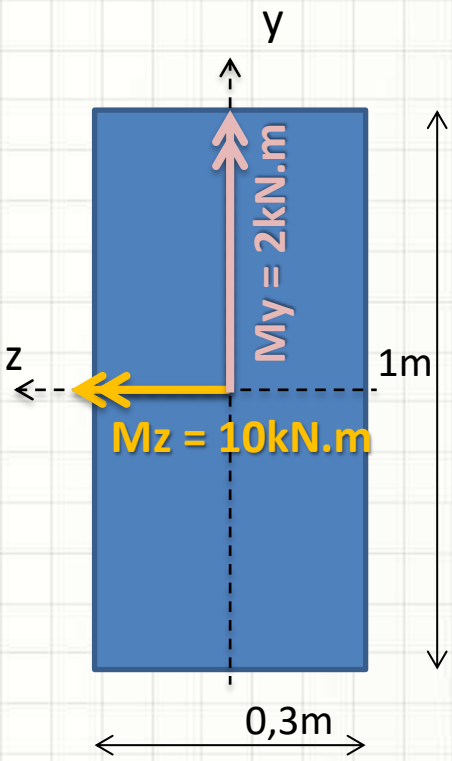




Exercício Pré-Intervalo

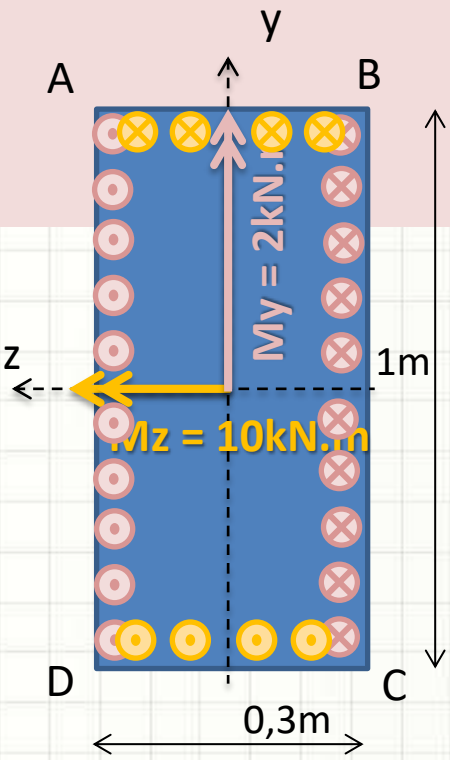
Exercício

- Qual a tensão de **compressão** máxima que surge?



Exercício

- Qual a tensão de **compressão** máxima que surge?



$$I_z = \frac{b.h^3}{12} = \frac{0,3.1^3}{12} \qquad \boxed{I_z = 0,025m^4}$$

$$I_y = \frac{b.h^3}{12} = \frac{1.0,3^3}{12} \qquad \boxed{I_y = 0,00225m^4}$$

$$\sigma_B = -\frac{M_y.c_z}{I_y} - \frac{M_z.c_y}{I_z} = -\frac{2000.0,15}{0,00225} - \frac{10000.0,5}{0,025}$$

$$\sigma_B = -133333 - 200000 = -333,33kPa$$

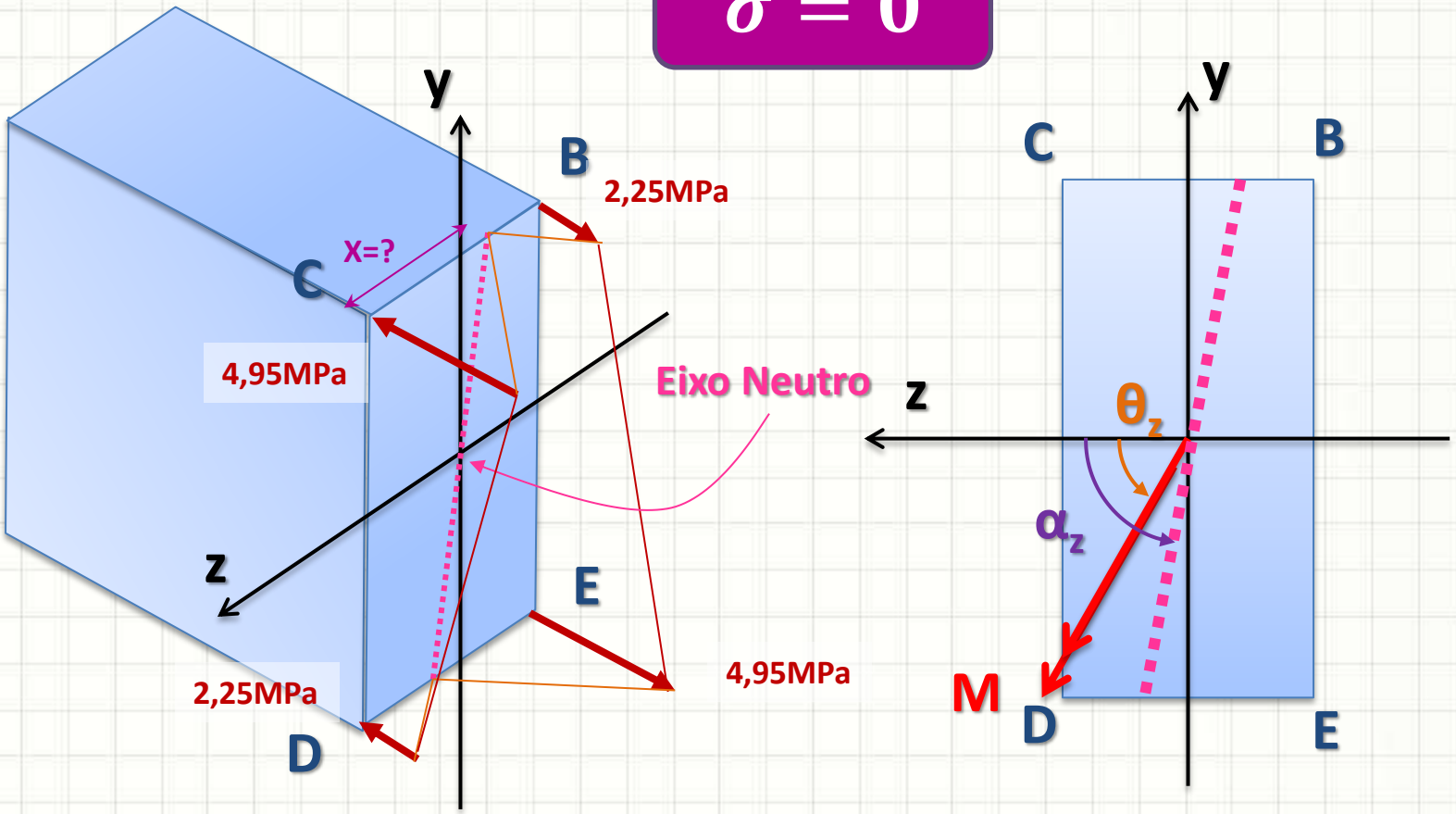
Pausa para o Café

Ângulo do Eixo Neutro

Eixo Neutro

- Se precisarmos saber onde é o eixo neutro...

$$\sigma = 0$$

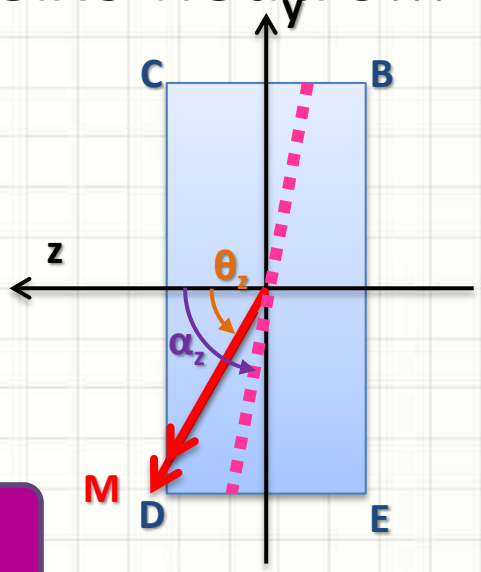


Eixo Neutro

- Se precisarmos saber onde é o eixo neutro...

$$\sigma = -\frac{M_z . y}{I_z} + \frac{M_y . z}{I_y}$$

$$\sigma = 0$$



$$0 = -\frac{M_z . y}{I_z} + \frac{M_y . z}{I_y}$$

$$\frac{y}{z} = \frac{M_y . I_z}{M_z . I_y}$$

$$M_y = M . \operatorname{sen} \theta_z$$

$$M_z = M . \cos \theta_z$$

- Ou seja...

$$\frac{y}{z} = \frac{I_z}{I_y} . \tan \theta_z$$

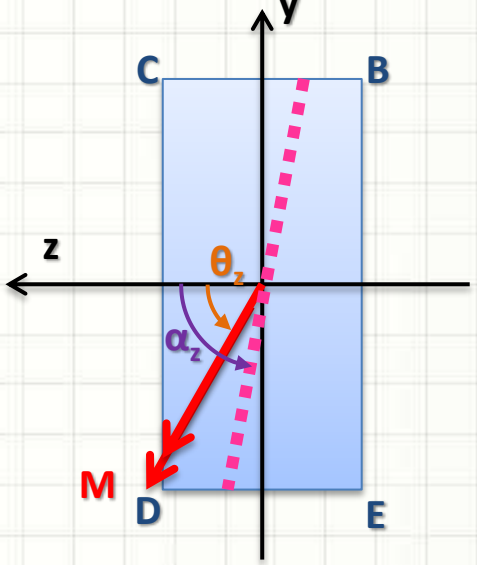
Eixo Neutro

- Se precisarmos saber onde é o eixo neutro...

$$\frac{y}{z} = \frac{I_z}{I_y} . \tan \theta_z$$

- Porém...
  - Para todo ponto no eixo neutro!

$$\frac{y}{z} = \tan \alpha_z$$



$$\tan \alpha_z = \frac{I_z}{I_y} . \tan \theta_z$$

$$\alpha_z = \operatorname{atan}\left(\frac{I_z}{I_y} . \tan \theta_z\right)$$

$\alpha \neq \theta$

Ou...

$$\alpha_y = \operatorname{atan}\left(\frac{I_y}{I_z} . \tan \theta_y\right)$$

Exemplo

- Calcule o Ângulo do Eixo Neutro

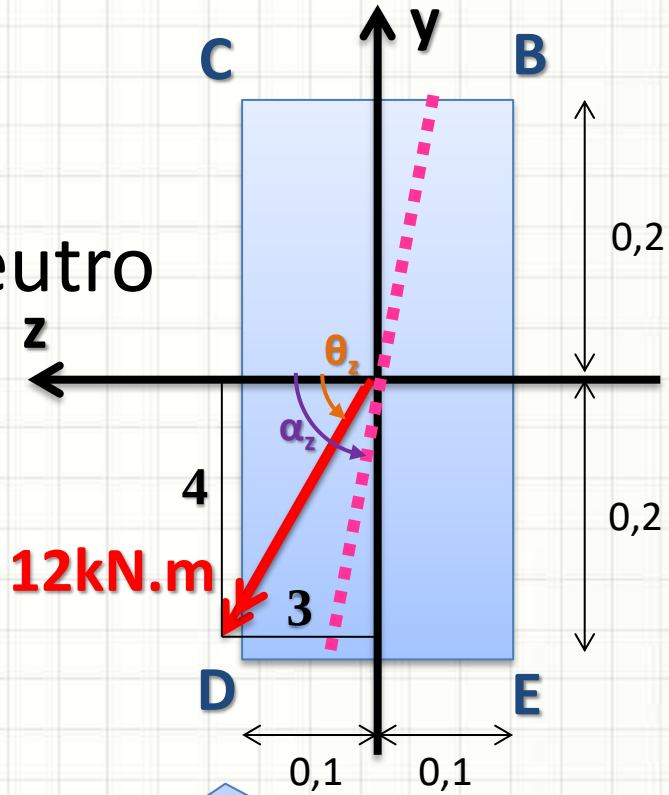
$$I_y = \frac{b.h^3}{12} = 0{,}000266\ldots m^4$$

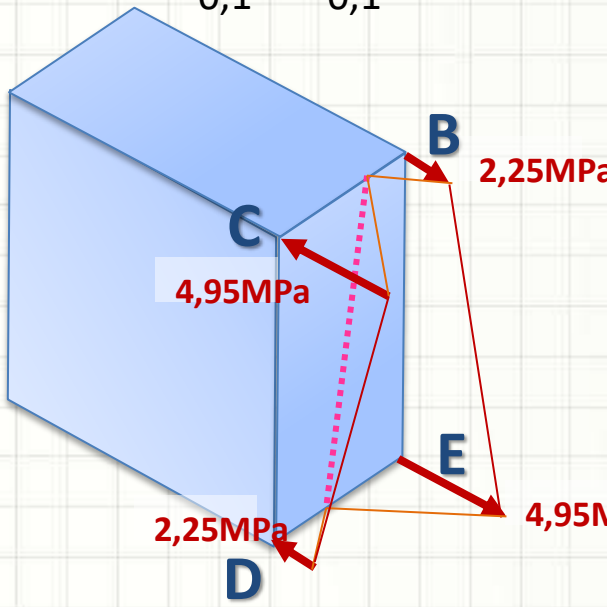
$$I_z = \frac{b.h^3}{12} = 0{,}0010666\ldots m^4$$

$$\alpha_z = \operatorname{atan}\left(\frac{I_z}{I_y}.\tan\theta_z\right)$$

$$\alpha_z = \operatorname{atan}\left(\frac{0{,}0010667}{0{,}0002667}.\frac{4}{3}\right)$$

$$\boxed{\alpha_z = 1{,}39rad = 79{,}4°}$$

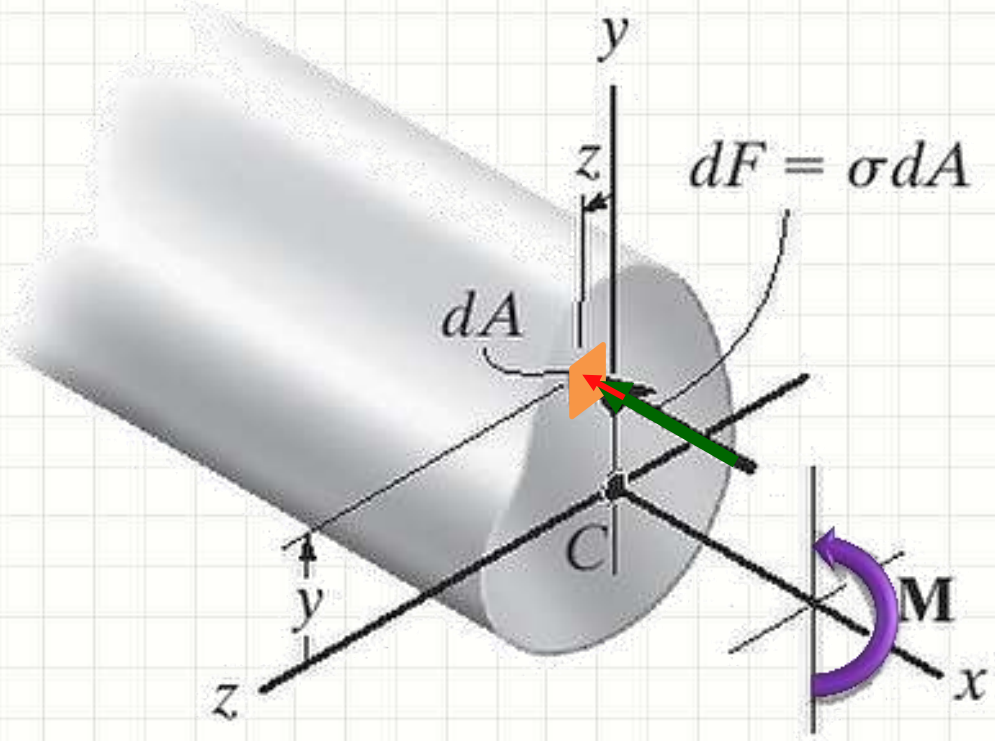




Flexão Assimétrica

Flexão Assimétrica

- Consideremos a seguinte seção assimétrica

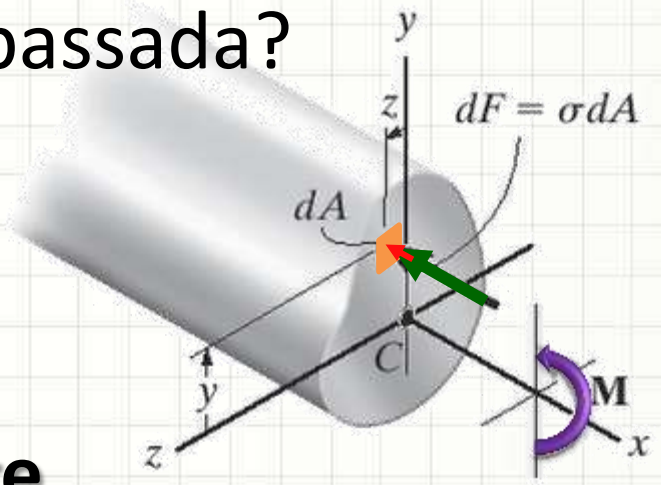


- M induz σ
- dF = σ . dA
- Equilíbrio?
  - $\sum F_x = 0$
  - $\sum M_z = M$
  - $\sum M_y = 0$

Flexão Assimétrica

- O que descobrimos na aula passada?
- **Z no eixo neutro garante**

$$\sum F_x = 0$$



- **A relação $\sigma_{máx} = \frac{M.c}{I}$ garante**

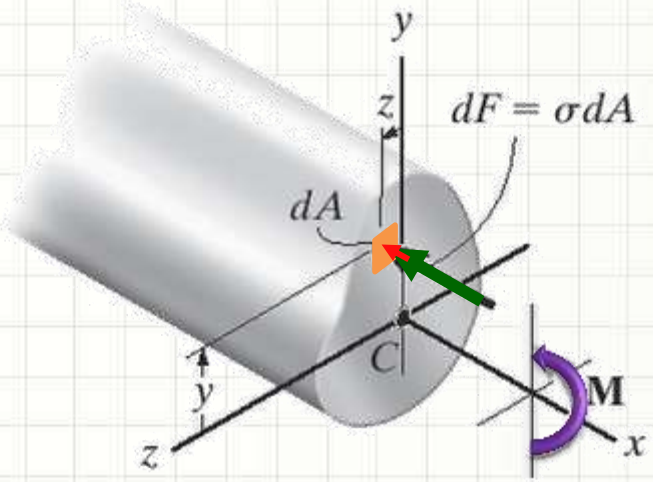
$$\sum M_z = M$$

- **Como garantir**

$$\sum M_y = 0$$?

Flexão Assimétrica

$$\sum M_y = 0 \rightarrow \int_A z.\sigma.dA = 0$$



- Mas... $\sigma = -\left(\frac{y}{c}\right).\sigma_{máx}$

$$\int_A -z.\left(\frac{y}{c}\right).\sigma_{máx}.dA = 0$$

- Resultará em...

Isso não tem como valer 0!

$$\frac{-\sigma_{máx}}{c}\int_A y.z.dA = 0$$

Quando isso vale 0?

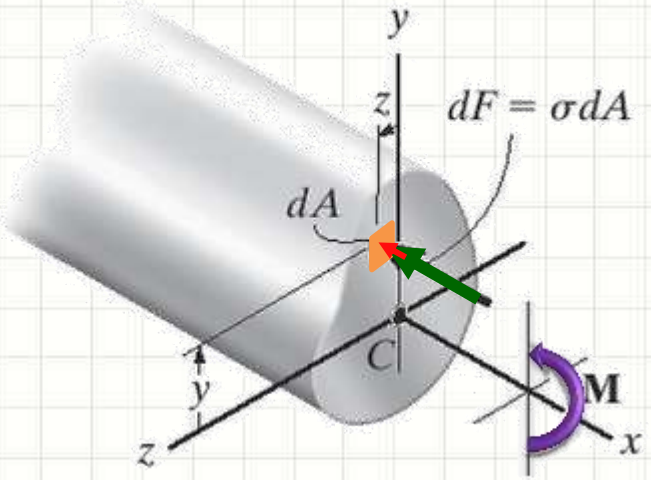
Produto de Inércia

Flexão Assimétrica

- Conclusão:
  - Momento é em torno de um dos eixos principais?

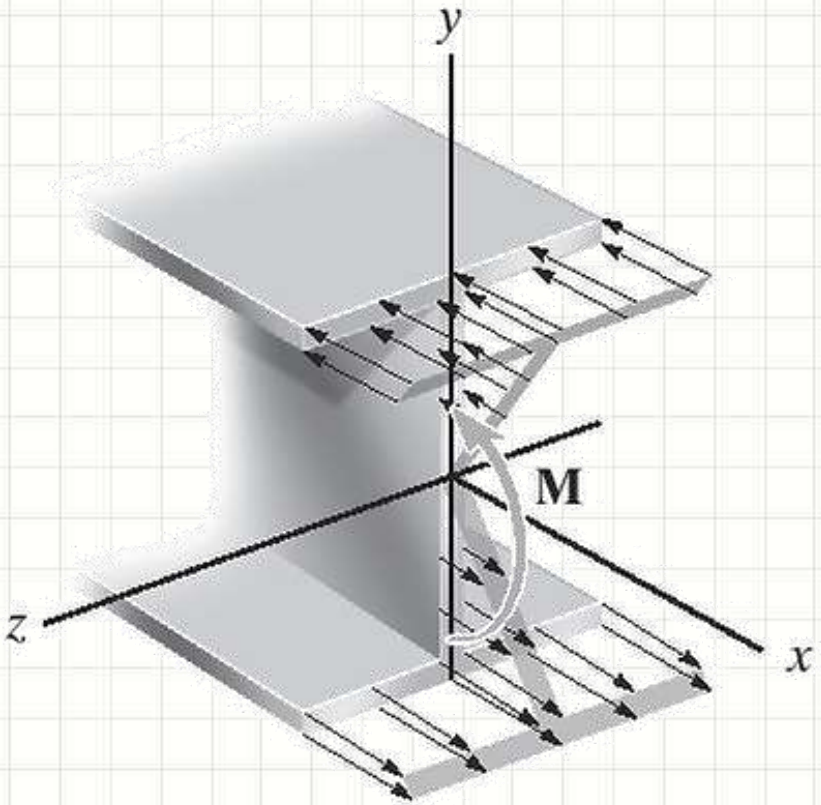
A teoria vale!

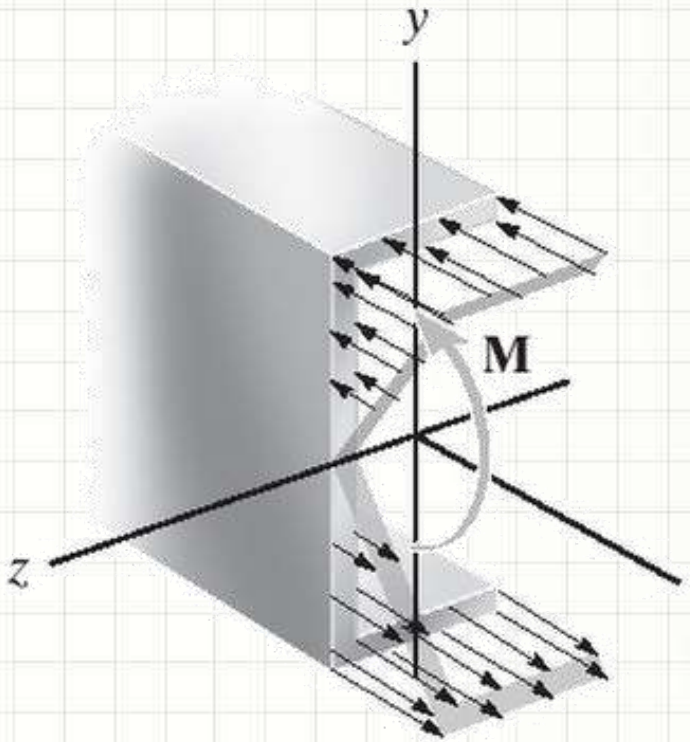
Simetria não importa!



Flexão Assimétrica

- Simetria ajuda...
  - Um dos eixos principais é o de simetria
  - O outro é perpendicular

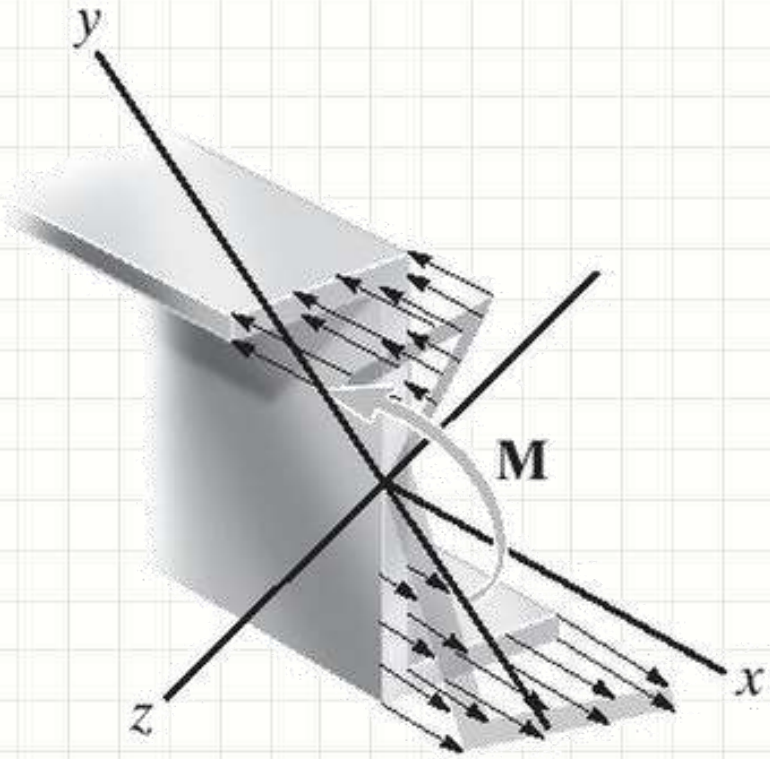


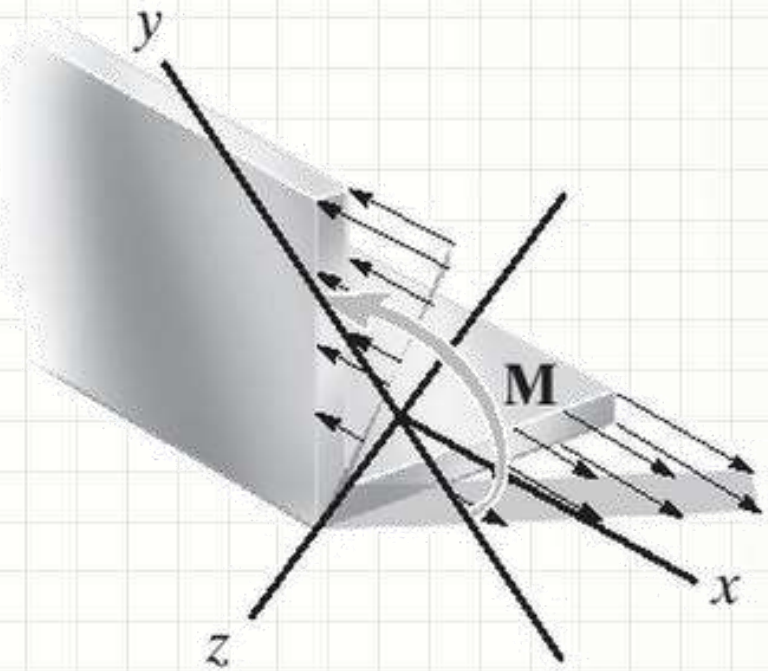


Flexão Assimétrica

- Se não há simetria...
  - Recorrer à fórmula
  - Ângulo dos Eixos Principais

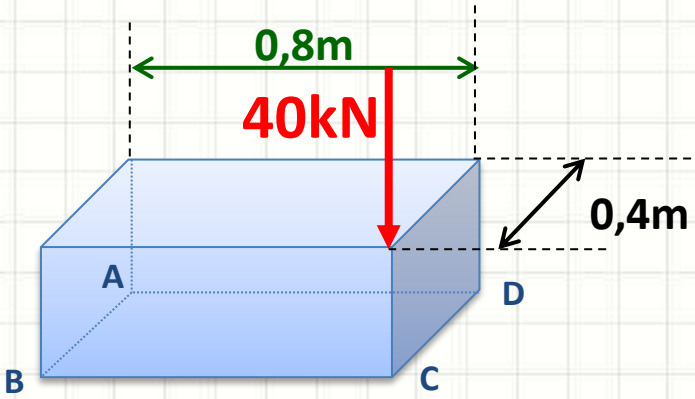
$$\theta_p = \frac{atan\left(\frac{2 \cdot I_{xy}}{I_y - I_x}\right)}{2}$$



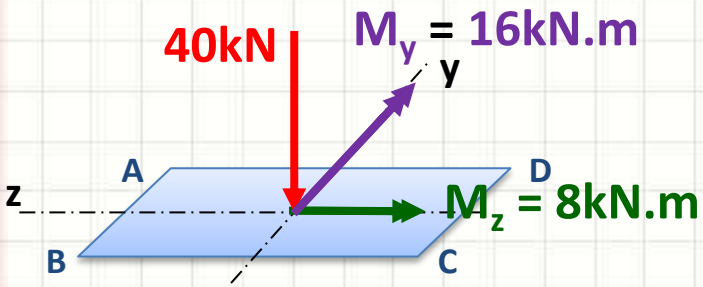


Exemplo de Flexo-Compressão Oblíqua

Exemplo

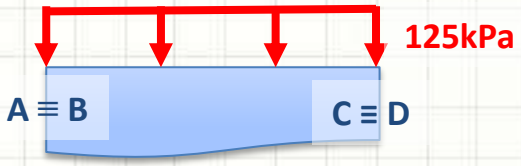


I) Cargas Atuantes



II) Força Normal

$$\sigma = \frac{F}{A} = \frac{40000}{0,8.0,04} = \mathbf{125kPa}$$



III) Momento Fletor em y

$$I_y = \frac{b.h^3}{12} = \frac{0,4.0,8^3}{12} \cong 1,71.10^{-2}\ m^4$$

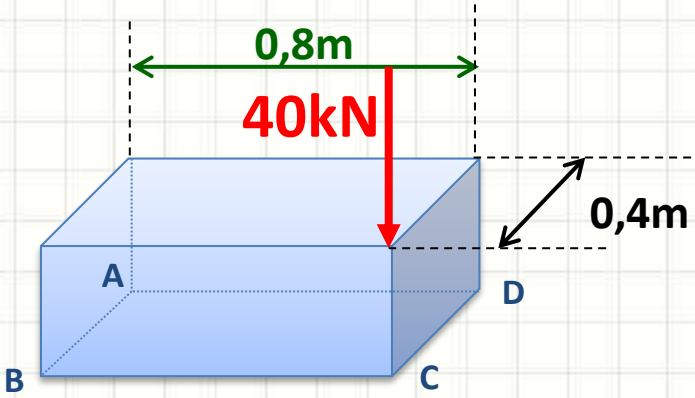
$$\sigma = \frac{M_y.z}{I_y} = \frac{16000.0,4}{1,71.10^{-2}} \cong \mathbf{375kPa}$$

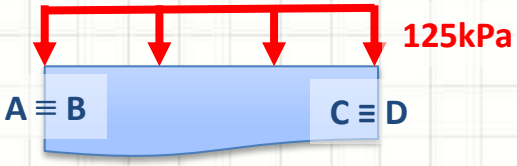
IV) Momento Fletor em z

$$I_z = \frac{b.h^3}{12} = \frac{0,8.0,4^3}{12} \cong 4,27.10^{-3}\ m^4$$

$$\sigma = \frac{M_z.y}{I_z} = \frac{8000.0,2}{4,27.10^{-3}} \cong \mathbf{375kPa}$$

Exemplo





Em A: $\sigma_A = -125 + 375 + 375 = 625\text{kPa}$

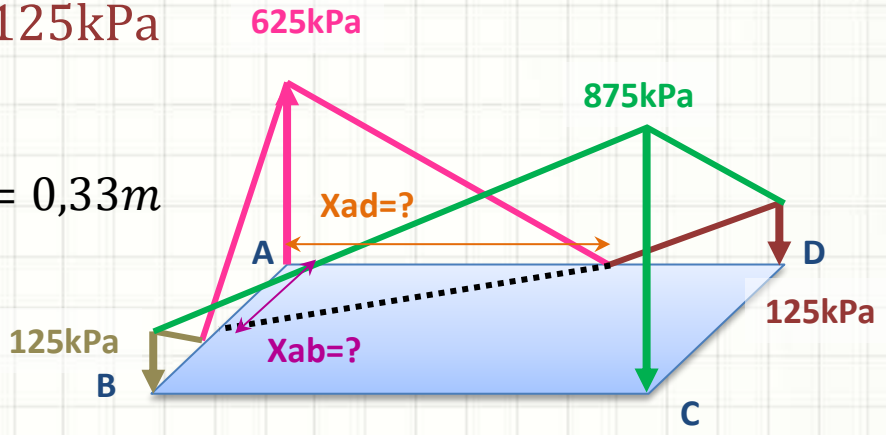
Em B: $\sigma_B = -125 + 375 - 375 = -125\text{kPa}$

Em C: $\sigma_C = -125 - 375 - 375 = -875\text{kPa}$

Em D: $\sigma_D = -125 - 375 + 375 = -125\text{kPa}$

$$\frac{\sigma_a + \sigma_b}{l} = \frac{\sigma_a}{x_{ab}} \qquad x_{ab} = \frac{\sigma_a . l}{\sigma_a + \sigma_b} = \frac{625.0{,}4}{750} = 0{,}33m$$

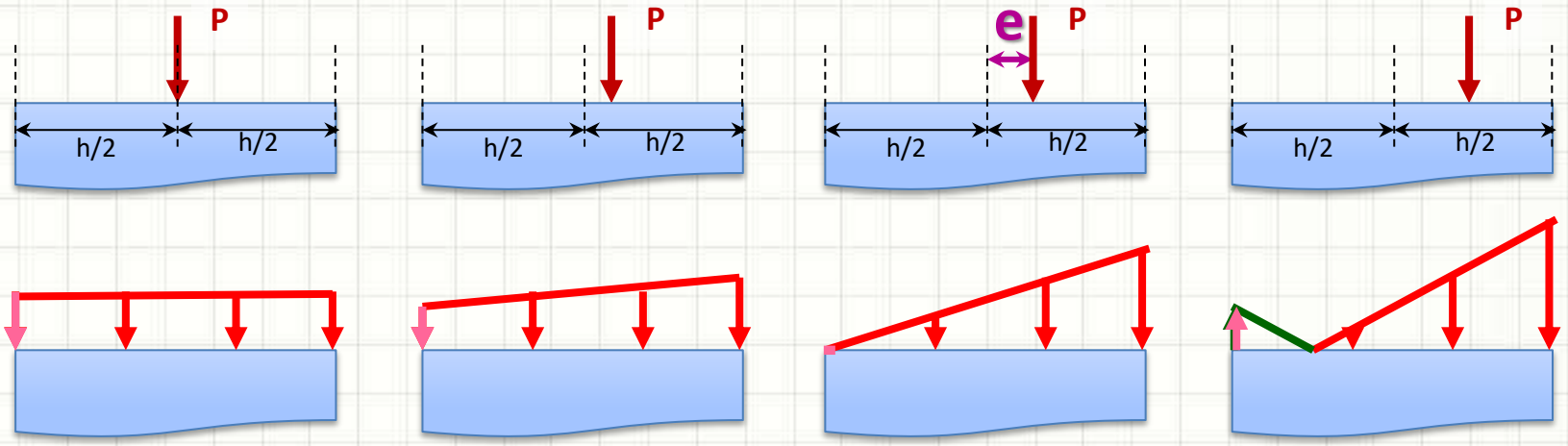
$$x_{ad} = \frac{\sigma_a . l}{\sigma_a + \sigma_d} = \frac{625.0{,}8}{750} = 0{,}66m$$



Núcleo Central de Inércia

Núcleo Central

- Imagine: tensões de uma carga central
  - O que acontece quando movemos a carga?



$$\sigma_{ext} = -\frac{P}{A} + \frac{M.c}{I} \qquad I = \frac{b.h^3}{12} = \frac{A.h^2}{12} \qquad M = P.e \qquad c = \frac{h}{2}$$

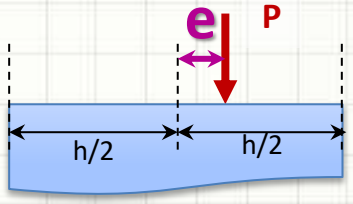
$$\sigma_{ext} = -\frac{P}{A} + \frac{P.e\ .h\ .12}{2\ .A.h^2} = -\frac{P}{A} + \frac{P}{A}.\frac{6.e}{h}$$

$$\boxed{\sigma_{ext} = -\frac{P}{A} + \frac{P}{A}.\frac{6.e}{h}}$$

Núcleo Central

- Queremos manter toda seção **comprimida**
  - Qual o maior valor de **e**?

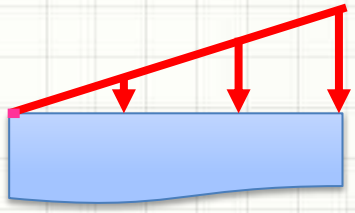
$$\sigma_{ext} = -\frac{P}{A} + \frac{P}{A}.\frac{6.e}{h}$$

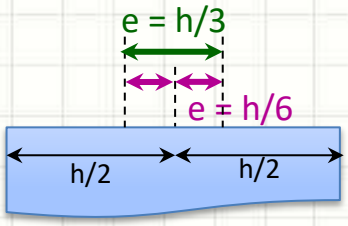


$$\sigma_{ext} = -\frac{P}{A} + \frac{P}{A}.\frac{6.e}{h} \leq 0$$

$$-\frac{P}{A} + \frac{P}{A}.\frac{6.e}{h} \leq 0 \rightarrow \frac{P}{A}.\frac{6.e}{h} \leq \frac{P}{A} \rightarrow \frac{6.e}{h} \leq 1$$

$$e \leq \frac{h}{6}$$





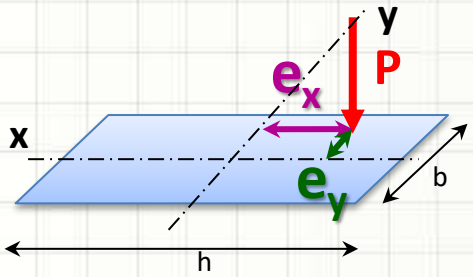
Núcleo Central

- Queremos manter toda seção **comprimida**
  - Considerando ambas as direções...

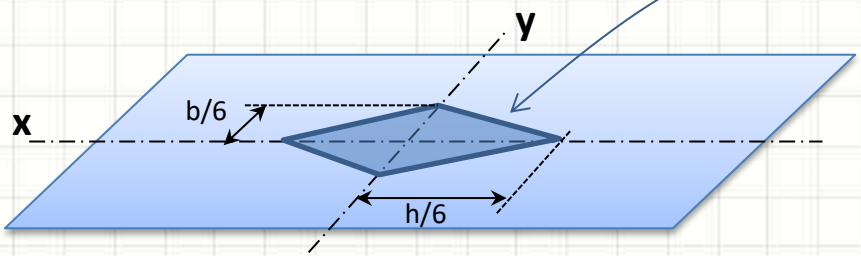
$$\sigma_{ext} = -\frac{P}{A} + \frac{P}{A}.\frac{6.e}{h}$$

$$\sigma_{ext} = -\frac{P}{A} + \frac{P}{A}.\frac{6.e_x}{h} + \frac{P}{A}.\frac{6.e_y}{b} \leq 0$$

$$\frac{6.e_x}{h} + \frac{6.e_y}{b} \leq 1$$

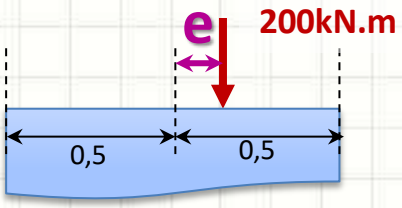


Núcleo Central



Exemplo

- No pilar abaixo, qual o maior valor de e para que o mesmo sofra apenas compressão?
  - Qual o valor da compressão máxima, sabendo que a seção é retangular de área 0,3m²?



$$e_{max} = \frac{h}{6} \qquad \boxed{e_{max} = 0{,}166m} \qquad I = \frac{b.h^3}{12} = \frac{0{,}3.1^3}{12} = 0{,}025m^4$$

$$\sigma_{ext} = -\frac{P}{A} - \frac{M.c}{I} = -\frac{200000}{0{,}3} - \frac{200000.0{,}166.0{,}5}{0{,}025} =$$

$$\sigma_{ext} = -666667 - 664000$$

$$\boxed{\sigma_{ext} \cong 1{,}33MPa}$$

Conclusões

Resumo

- Flexões compostas podem...
  - Ser decompostas para tratamento...
  - ...considerando-se os eixos principais
- Tensão máxima: por superposição de efeitos
- Ângulo da LN ≠ Ângulo do momento oblíquo
- **Exercitar: Exercícios Hibbeler**

- Exercitar para a avaliação!

Para Treinar

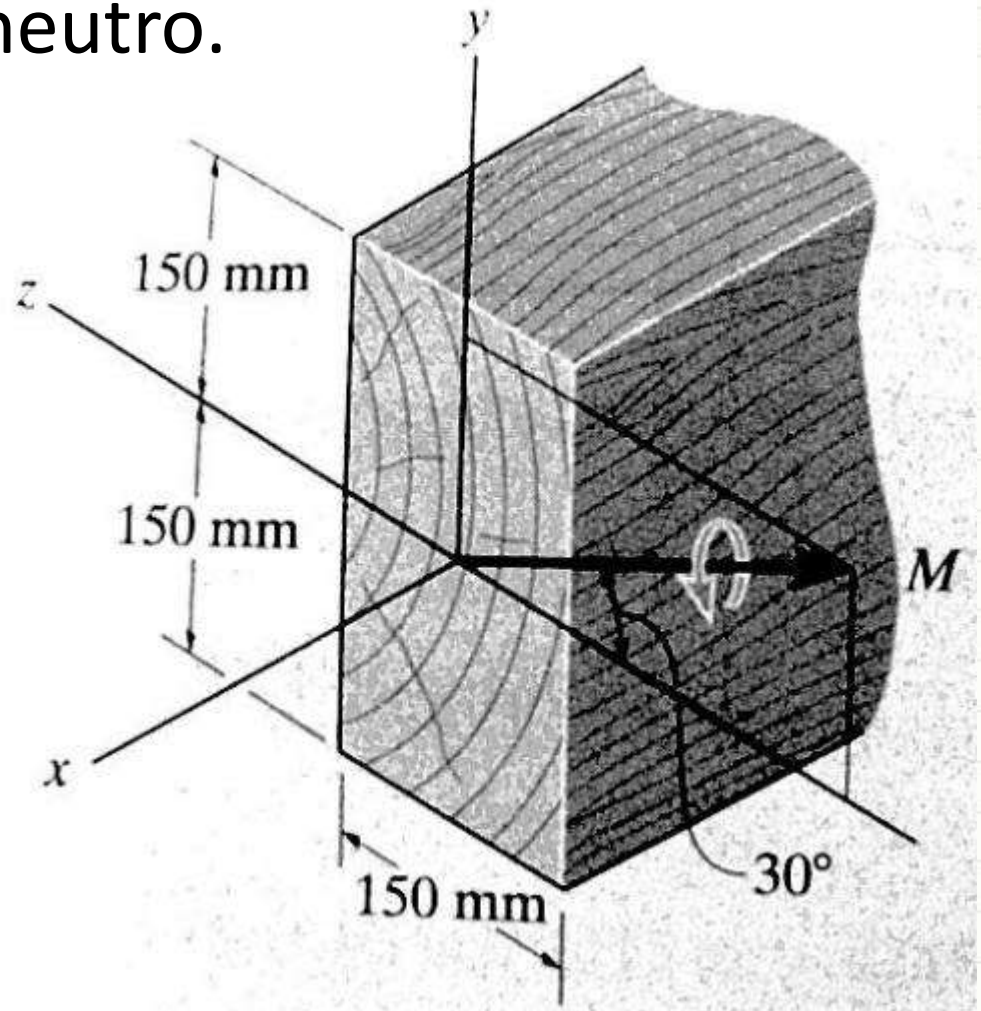
Para Treinar em Casa

- Mínimos:
  - Exercícios 6.104, 6.107, 8.20 e 8.21
- Extras:
  - Exercícios 6.103, 6.105, 8.26 e 8.60

EXERCÍCIO NO SAVA

Exercício – Entrega Individual

- Considerando M=3,5kN.m, calcule o $\sigma_{máx}$ e a direção do eixo neutro.

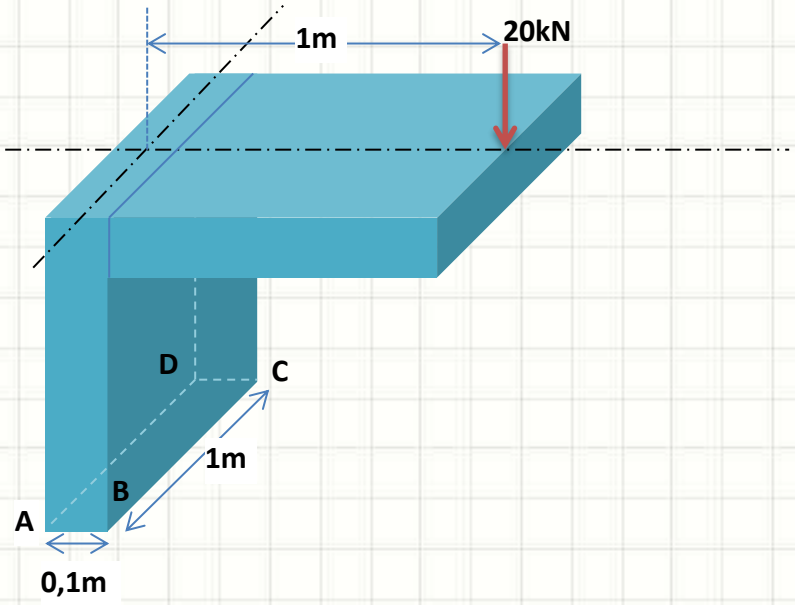


PERGUNTAS?

Exercício em Sala

Exercício – Individual, para **Agora**!

- Calcule a tensão máxima de compressão na base do pilar ABCD; ignore o peso próprio da estrutura.



$$\sigma_{max} \cong 12,2MPa$$